THEORY OF SOLID AND BRACED ELASTIC ARCHES

APPLIED TO

ARCH BRIDGES AND ROOFS IN IRON, WOOD, CONCRETE OR OTHER MATERIAL.

GRAPHICAL ANALYSIS.

BY

WM. CAIN, C. E.

REPRINTED FROM VAN NOSTRAND'S MAGAZINE.

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PREFACE.

THE analytical treatment of solid and braced arches is very complex, and does not admit of the same generality in the deductions as the graphical method. It is hoped that the latter, as now presented, will be found simple and practical, at least to the student of Graphical Statics, and may aid materially in a proper comprehension of the theory of arches of every description.

The subject has been logically deduced, assuming only an elementary knowledge of statics and of the theory of elasticity. The best graphical analysis heretofore presented is that of Prof. H. T. Eddy in his *New Constructions in Graphical Statics*, which is referred to concerning very flat arches, the only kind treated of in his constructions. The methods given in this book apply to

any style of arch, loaded in any manner; and it is claimed that the treatment is the first thorough and general one that the solid arch has ever received.

Much that is new will be found in what follows, notably, the exact solution of the voussoir arch in certain cases, the first correct graphical determination of temperature strains and of the arch hinged outside of the center line.

The notation of the equilibrium polygons, too, will be appreciated, as it tends to avoid mistake.

It is hoped that the book will be found useful to the practitioner and of interest to the student of Graphical Statics, containing as it does the most important applications of the equilibrium polygon.

THE AUTHOR.

THEORY OF

Solid and Braced Elastic Arches.

GENERAL OBSERVATIONS AND THEOREMS.

1. The term *solid arch* is applied, in what follows, to arches having a continuous web connecting the flanges; the term *braced arch*, to such as are braced between the flanges by the usual struts and ties, forming any pattern of open web. As contradistinguished from the voussoir arch, the solid or braced arch is capable of supplying tensile resistances, at any ideal section, when needed; though as all arches are composed of elastic materials the term *elastic arch* is applicable to any one of them. When the solid arch is hinged at one or more points (not exceeding three) there are of course no tensile forces exerted at the hinged joints, so that the hinged joints, if any, must be excepted in the above definition.

If the arch has more than three hinged joints, it is no longer stable, except as a suspension bridge. In the voussoir arch, composed of many stones laid flat against each other, the joints are not hinged; in fact, if the arch ring is of such proportions, that only compressive forces are exerted at each actual joint, the treatment may fall under that pertaining to solid arches, as was first suggested in the preceding paper on *Voussoir Arches*, VAN NOSTRAND'S MAGAZINE, Vol. XX, article 27, remark. This connection between solid and voussoir arches, will be more fully developed in the present paper. The aim has been, in what follows, to treat solid and braced arches in the most general manner; the graphical method offering great facilities in making the discussion very comprehensive, and at the same time simple and obvious. As is well known, the strictly analytical treatment of this subject, especially in the applications, is tedious in the extreme. For the best analytical discussion, the reader is referred to that of Winkler, given in

Du Bois' very comprehensive *Graphical Statics.* It is hoped that the purely graphical solution following may be comprehended by many, who posess only a knowledge of the composition and resolution of forces and the principles of moments as taught in mechanics. To this end a brief review of the eqilibrium polygon and some of its properties seems necessary, after which the theory in question will be entered upon.

The Graphical analysis has received attention from Wm. Bell (*Rigid Arches, &c.*, VAN NOSTRAND'S MAGAZINE, Vol. VIII); Prof. Chas. E. Greene (in *Engineering News*, 1877), and Prof. H. T. Eddy in *Researches in Graphical Statics*; much of the latter treatise having first appeared in VAN NOSTRAND'S MAGAZINE for 1877.

The method adopted by Prof. Eddy constitutes a marked advance in precision of treatment, rendering possible the ready and exact location of the true curve of pressures by a systematic method, in place of the method of innumerable trials, heretofore resorted to. It is largely drawn

upon in what follows. In order to render the solution as exact as the graphical method will admit of, the few principles of the theory of elasticity that are needed will also be demonstrated.

2. *Equilibrium Polygon.*—Let the parallel forces (Fig. 1) acting through r_1, p_1, p_2, p_3 and r_2 be in equilibrium, their intensities being represented, to the scale of force, by the lines R_1, P_1, P_2, P_3 and R_2 on the left, drawn parallel to the forces.

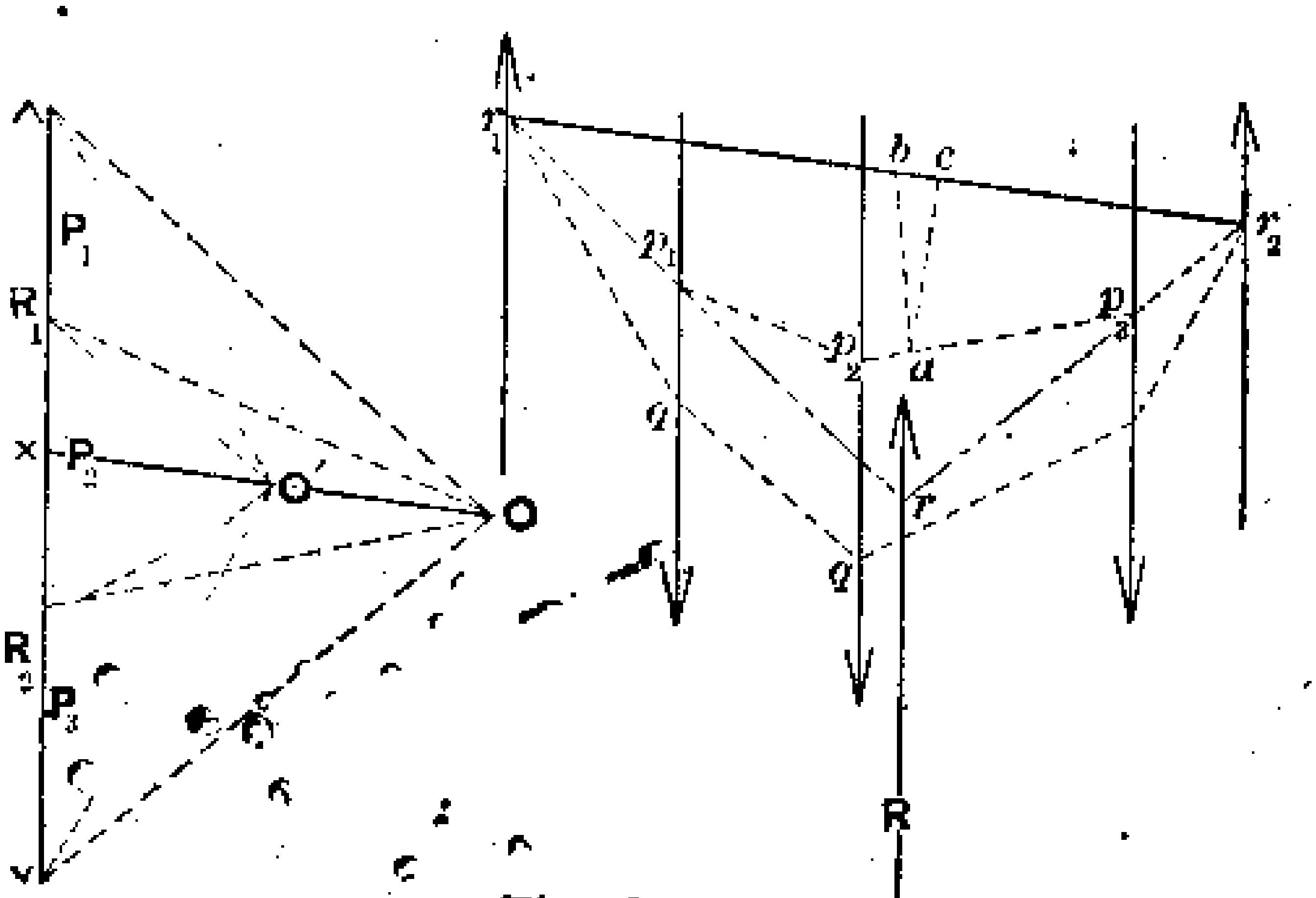


Fig. 1.

From any point O, not on the force line P, draw the dotted rays to the extremities of the lines representing

P_1, P_2 . . . . Now commencing at any point r_1 of the first force on the left, draw $r_1 p_1 \parallel$ ray R_1P_1, then $p_1 p_2 \parallel$ ray P_1P_2, $p_2 p_3 \parallel$ ray P_2P_3 and $p_3 r_2 \parallel$ ray P_3R_2; meaning by ray R_1P_1, P_1P_2, . . ., the rays from O included between the letters mentioned.

Now it will be found that the line $r_1 r_2$ is parallel to the ray R_1R_2 or OO′ as may easily be proved from mechanical principles. Thus: conceive applied along each of the lines $r_1 p_1$, $p_1 p_2$, $p_2 p_3$ and $p_3 r_2$, two opposed forces, each equal to the force that is measured by the length of the ray to which the line is parallel. This does not disturb equilibrium, but it adds at p_1, two forces equal to the rays P_1R_1 and P_1P_2 measured to the scale of force. Similarly at p_2 and p_3 are added the forces represented by the rays P_1P_2, P_2P_3 and P_2P_3, P_3R_2, so that complete equilibrium is established at p_1, p_2 and p_3, since there are three forces at each point proportional to the sides of a triangle parallel to their directions, as we see from the force diagram on the left. Moreover,

since the P's act downwards, the applied forces at each vertex, $p_1, p_2 \ldots$, must act *away* from that vertex, as we see by following round the triangle of forces for each vertex p.

Now as the primary forces are in equilibrium, we must likewise have complete equilibrium at both r_1 and r_2 from the original forces R_1 and R_2 and the remaining forces applied along the lines $r_1 p_1, p_3 r_2$.

Now in order to have separate equilibrium at both r_1 and r_2 we must apply two opposed forces at r_1 and r_2, each equal to the resultant of R_1 and ray R_1P_1, or of R_2 and ray R_2P_3, (*i.e.* equal to the ray R_1R_2) *which forces balance only when this ray* R_1R_2 *is parallel to* r_1r_2.

Hence, if the forces P_1, P_2, P_3, acting at p_1, p_2, p_3, are given, and it is required to find the forces at r_1 and r_2 (similar to the reactions of a loaded girder) we lay off the forces P_1, $P_2 \ldots$, in order, parallel to their directions; then starting at r_1 or on a vertical through r_1, we draw $r_1p_1 \parallel$ ray R_1P_1, $p_1p_2 \parallel$ ray P_1P_2 etc., to r_2;

then a line OO′ drawn parallel to r_1r_2 will divide the force line P into the two reactions. R_1 and R_2, for now at each point, r_1, p_1, . . . , on supplying the imaginary forces we have equilibrium. The line r_1r_2 is called the *closing line*, and the polygon $r_1\,p_1$ . . . . r_2 the *equilibrium polygon.*

Further; if we prolong the sides $r_1\,p_1$ and $r_2\,p_3$ to intersection r, the resultant $R = R_1 + R_2$ acts through r, and is of course parallel to the original forces. This is evident if we consider only the forces at r_1, r_2 acting upwards, and R acting downwards at r, we see that on supplying the opposed forces along the lines rr_1, r_1r, r_2r equal to rays P_1R_1, R_1R_2, P_3R_2 that the system is in equilibrium.

3. We shall repeatedly have to solve this problem: given the forces at $p_1, p_2 \ldots$, to find their resultant R. We have then simply, from some point p_1 on the first force, to draw p_1p_2, ∥ ray P_1P_2, &c.; from the last point p_3 so found, draw p_3r ∥ ray P_3R to intersection r with p_1r ∥ ray P_1R. It is

seen that on supplying the imaginary forces that the system is in equilibrium at each vertex p_1, p_2, . . . p_3, r. The force R at r, acting in an opposite direction to that which causes equilibrium, is the resultant of the forces at p_1, p_2 . . . in position, direction and magnitude.

4. Again, let us consider the original forces at r_1, p_1, p_2, p_3 and r_2 in conjunction with the forces applied along the sides of the equilibrium polygon $r_1\, p_1\, p_2 \ldots r_2$ that cause equilibrium at each vertex. Let us suppose a vertical section made at any point a and that the right part abr_2 is removed. Now equilibrium still obtains at the apices r_1, p_1 and p_2. Now for any system of forces in equilibrium, the sum of their moments about any point in their plane is zero. The forces along $r_1\, p_1$ and $p_1\, p_2$ balance, leaving the forces R_1, P_1, P_2 (whose moment about a we shall call M), the force along $p_2\, p_3$ and that along $r_1 b$ which is represented by C. Taking moments about a,

$$M - C.\overline{ac} = O$$

Call the perpendicular from the pole O to the force line $P_1 \ldots P_3$ *the pole-distance*$=$H.

Now ac the perpendicular from a upon $r_1 r_2 = \overline{ab}.$ cos. bac $\therefore$ C. $\overline{ac} =$ C. cos. $bac \times ab$ $=$ H. $\overline{ab}$, as is seen from the left figure or force diagram

$$\therefore \text{M} = \text{H}.ab = \text{H}y.$$

Or the moment of the vertical forces to one side of any point a is equal to the pole distance, H measured to the scale of force multiplied by the ordinate of the equilibrium polygon at that point, to the scale of distance. M is the same taken about any point in the line ab; hence is measured by H.$\overline{ab}$ for any point in the same vertical.

It is this property of the equilibrium polygon that we shall find of special utility in treating solid arches.

As the above result must be true whenever the pole O is placed, we see that for any number of equilibrium polygons the product Hy is constant; *whence a decrease of H involves an increase in y in the same ratio, and the reverse.*

5. If we desire to construct a new equilibrium polygon, with the same closing line r_1r_2 as the first, the new pole O′ must be taken on the ray R_1R_2, since this ray must be parallel to the closing line. The equilibrium polygon is constructed as before, as shown in the figure. If some other pole had been chosen, lying in a vertical line through O′, on drawing the equilibrium polygon through r_1 as before, the point r_2 will be found above or below its present position, but the ordinates y remain the same. If these ordinates are taken in dividers and laid off from the present position of r_1r_2, the equilibrium polygon qq ... can be located without the necessity of drawing it directly.

6. If we denote the horizontal distances of the forces R_1, P_1, P_2, acting at r_1, p_1, p_2, from E by x_0, x_1, x_2, respectively, we have just found that,

$$R_1x_0 - (P_1x_1 + P_2x_2) = H.\overline{ab} = H.y.$$

Produce $\overline{ba}$ to intersection r (it happens to be) with r_1p_1 produced. Now the triangle r_1rb is similar to the one with

vertex O and base R_1; $\therefore$ H: R_1:: x_0: $\overline{br}$, or $R_1x_0 = H.\overline{br}$. Substituting this value above, we have,

$$P_1x_1 + P_2x_2 = H.\overline{ar}.$$

From this equation it is evident that we have a ready means of finding the sum of a series of products, of the type Px, in the equilibrium polygon. We have only to draw verticals through points p_1, p_2, whose horizontal distances, to some scale, from the vertical $\overline{ab}$, are x_1,x_2, respectively; then on laying off to some scale, (the same as before if desired) the quantities P_1,P_2, along the line P_1P_2 and constructing an equilibrium polygon through any point r_1, by drawing $r_1p_1 \parallel$ ray R_1P_1 $p_1p_2 \parallel$ ray P_1P_2, $p_2p_3 \parallel$ ray P_2P_3 to intersection a with the vertical $\overline{ab}$, and the closing line r_1a in this case, we find $P_1x_1 + P_2x_2 = H.\overline{ar}$ as before; H being measured to the scale of the P's, and $\overline{ar}$ to the scale of the x's. The principle evidently applies to any number of supposed forces lying to the left of ab. It is evident that the principle is true however near r_1 is taken to p_1. The construction is

simplified by causing r_1 and p_1 to coincide so that p_1a will be the closing line. The proof is easily extended to the case when the forces are laid off on $P_1 \ldots P_3$ in any order, and some of them act in opposite directions.

7. The following is a simple geometrical proof that involves the important problem of quadratures. Suppose it required to find the numerical value of the sum of the products $(P_1x_1 + P_2x_2 + P_3x_3 + P_4x_4)$, P_1, x_1, etc., denoting any numerical values whatsoever.

Draw the vertical aa and the horizontal xx (Fig. 2). Through the points x_1, x_2, . . distant x_1, x_2, . . to scale, from aa draw vertical lines, x_1p_1, etc.

Next lay off to some scale (the same scale used for the x's if preferred) P_1, P_2, . . ., in order on a vertical line P_1P_4 and from some point O, draw the rays to the extremities of P_1, P_2, . . ., and denote any ray by the letters enclosing it. Then starting at some point a, draw $ap_1 \parallel$ ray AP_1 to p_1 in the first vertical, then draw $p_1p_2 \parallel$ ray P_1P_2, to intersection

p_2, with the vertical p_2, then $p_2 p_3 \parallel$ ray P_2P_3, $p_3\, p_4 \parallel$ ray P_3P_4 and $p_4\, a_4 \parallel P_4A_4$ to intersection a_4 with the vertical aa. Then denoting by H the perpendicular from O upon the line P_1P_4, measured to the scale of the P's, and measuring $aa_4 = v$, to the scale of the x's, we have

$$P_1x_1 + P_2x_2 + P_3x_3 + P_4x_4 = H.v$$

The proof is easy. Extend the lines $p_1\, p_2$, $p_2\, p_3$, . . . to intersection a_1, a_2, . . . with aa; then each triangle, $p_1\, aa_1$, $p_2a_1a_2$, . . . is similar to a triangle in the figure on the left, the sides being parallel; hence the ratio of base to altitude is the same for both

$$\therefore H : P_1 :: x_1 : \overline{aa_1} \quad \therefore P_1x_1 = H.\overline{aa_1}$$
$$\therefore H : P_2 :: x_2 : \overline{a_1a_2} \quad \therefore P_2x_2 = H.\overline{a_1a_2}$$
. . . .

Adding the equations, we have

$$P_1x_1 + P_2x_2 + \ldots = H.\overline{aa_4} = H.v$$

as stated above. This is the general principle involved in Culmann's "Summation Polygon." The notation employed

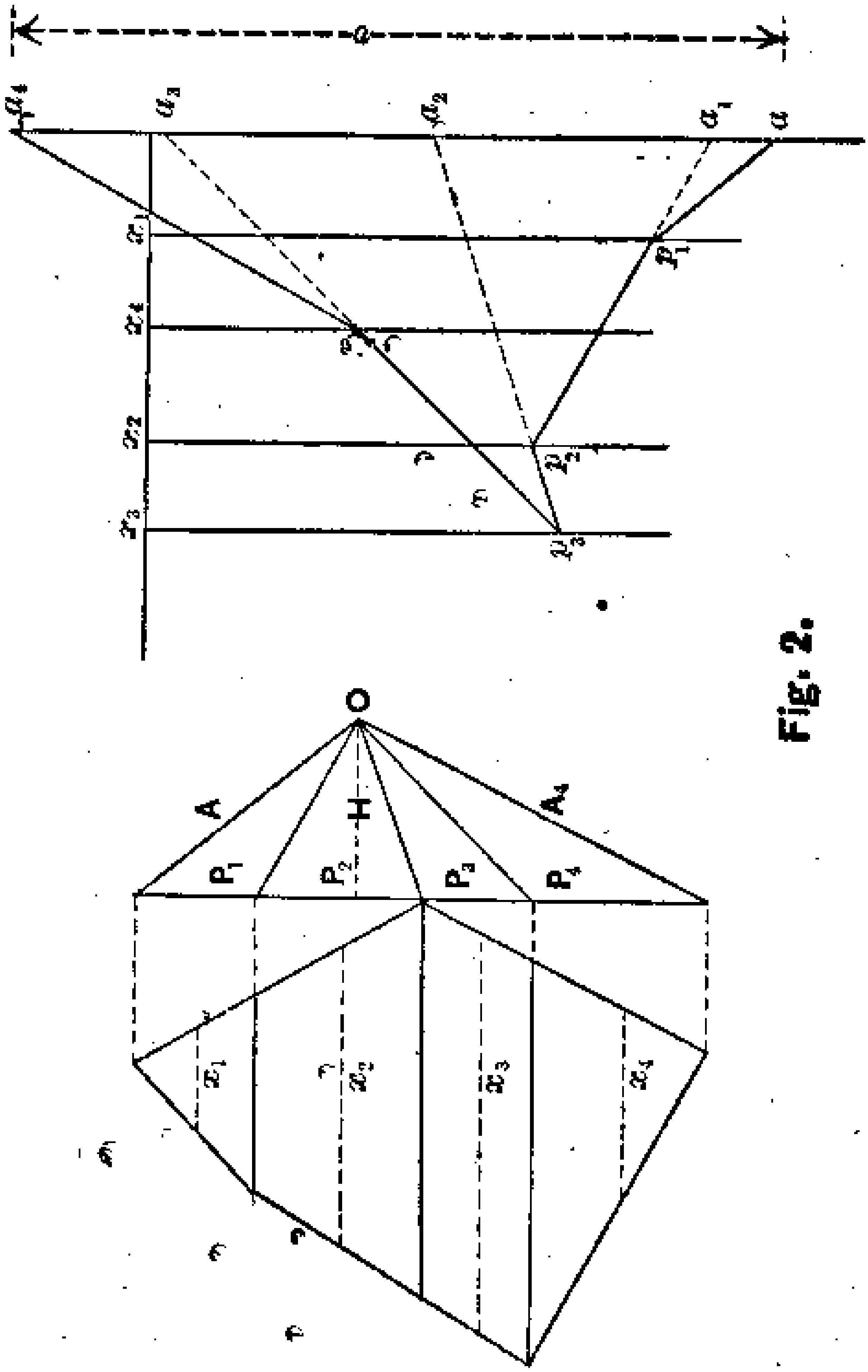


Fig. 2.

above seems to forbid the possibility of a mistake in construction.

On applying this rule to the products

$(P_1x_1+P_2x_2)$ and $(P_3x_3+P_4x_4)$ separately, beginning the construction for the last at a_2, we see that $(P_1x_1+P_2x_2)-(P_3x_3+P_4x_4)$ $=H(aa_2-a_2a_4)$, a principle that we shall find of great utility.

To apply the above principle to *areas*, as the one shown on the left, draw through each corner of the plat, lines $\perp P_1P_4$, thus dividing the figure into triangles and trapezoids. Take the horizontal medial lines x_1, x_2, . . . of each triangle and trapezoid in dividers, and lay off on line xx as above. The lines P_1, P_2, . . . are the altitudes of the triangles or trapezoids; so that by choosing a pole O at a distance H (an even number is best) from P_1P_4, and proceeding as above we find $(P_1x_1+P_2x_2+\ldots)=H.v$ $=$area of figure. In this case H and v are measured to the same scale.

If another plat is made alongside of the one shown, the area of both may be found at one operation; the distances x_1, x_2, . . will now denote the sum of the medial lines of the corresponding trapezoids of both figures. Where the same

area has a slice taken out of it, this principle will be needed.*

It is generally convenient to choose H about the mean width of the plot, whence v will be the mean height about, and the construction to the right is kept within bounds.

8. Suppose it required to find $(P_1x_1^2 + P_2x_2^2 + \ldots)$ or the *moment of inertia* of the forces $P_1, P_2, \ldots$ acting at $x_1, x_2 \ldots$ about the axis aa. After making the above construction, regard the segments $aa_1, a_1a_2, \ldots$ as forces applied at $x_1, x_2, \ldots$, and make a construction similar to the above with a pole distance H′, and denote the segments cut off on the line aa_1, by $bb_1, b_1b_2, \ldots$. Then we have, as before,

$$\overline{aa_1} \cdot x_1 = H' \cdot \overline{bb_1},\ \overline{a_1a_2} \cdot x_2 = H' \cdot \overline{b_1b_2}, \text{ etc.}$$

* I have recently used the above method in checking numerical computations, and find that for small tracts of land (80 acres say), the difference between the graphical and numerical methods was nearly zero, for two 100 acre tracks, $\frac{1}{10}$ of an acre difference; for a 250 acre tract, 3.1 acres; for a 400 acre tract 2.2 acres. Scale used was 500 feet=1 inch. The above is only intended to give some idea of the accuracy of the method, using only ordinary care.

Multiplying both sides by $\overline{H}$ and adding, noting that by art. 7 $H.\overline{aa_1}=P_1x_1$, etc., we find,

$$(P_1x_1^2+P_2x_2^2+\ldots)=H'.H.\overline{bb}_4.$$

For a full discussion of the elements of Graphical Statics, the reader is referred to DuBois' *Graphical Statics*, and Eddy's *Researches in Graphical Statics*. We have already nearly all the principles of the method that we shall use in this treatise, and shall now proceed to their application in the case of the arch.

THEORY OF ELASTICITY.

9. Let Fig. 3 represent a portion of a *solid arch with parallel flanges*, whose neutral axis, when ordinary flexure alone is considered, is *nn*.

Consider the part, whose length measured along *nn* is *s*, included between the two normal planes that make an angle a before strain and a' after strain. Let R be the resultant of the external forces at the middle of the part taken. At the middle of *s* on *nn* conceive two equal

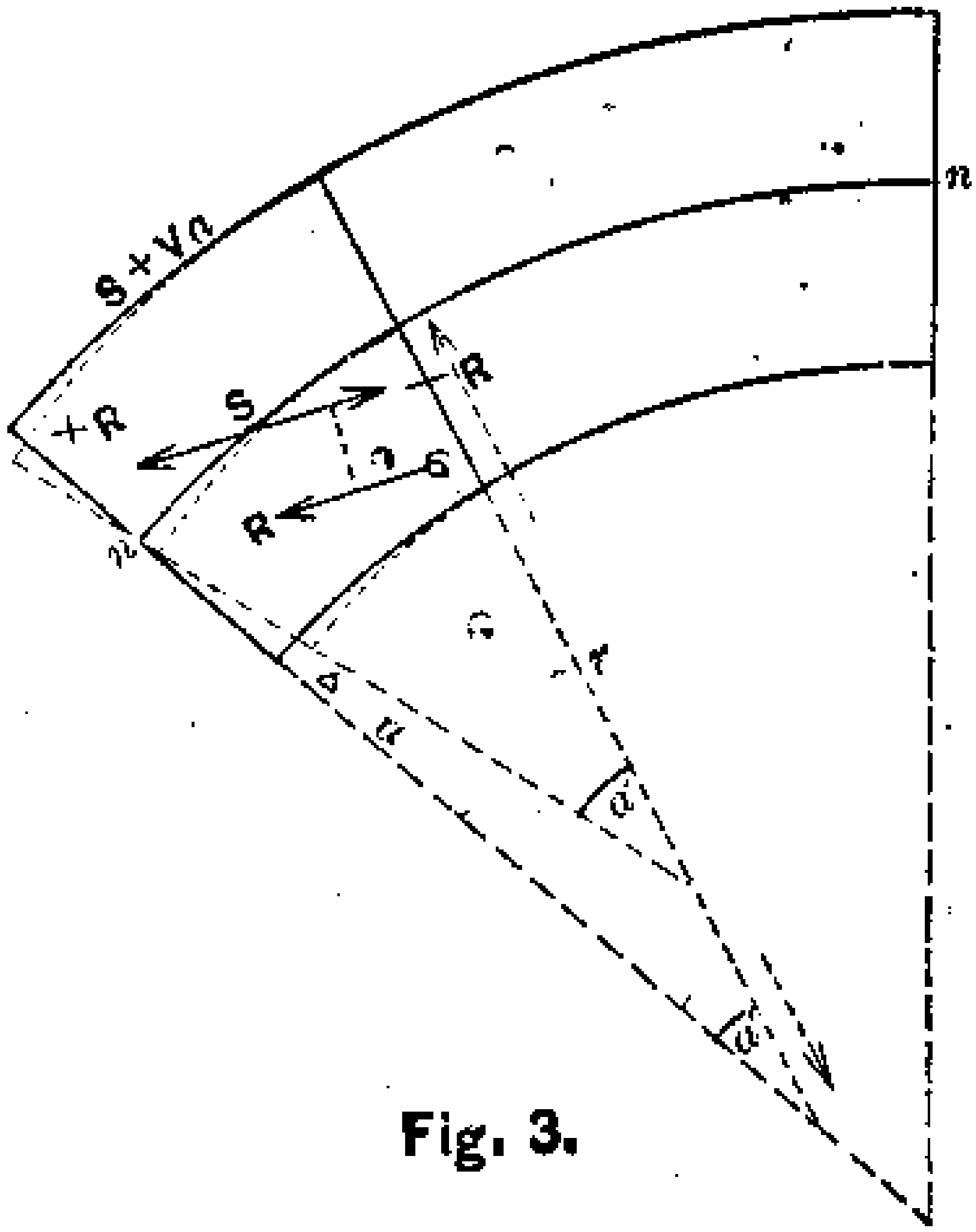


Fig. 3.

opposed forces $+R$, $-R$, each equal to R. The force R is thus equivalent to a couple $\overline{RR}$ and a force $+R$ at the middle of s on nn. This latter force may be decomposed into two components T and N tangential and normal to nn at the point of action. The forces T on each part produce a shortening of the entire arch and its effect may be considered by itself, as we shall see. The force N is similar to the shearing strain in a straight

beam (see art. 82). The moment RR is principally effective in changing the curvature of the arch, and it will alone be considered in this connection at present. The deformation caused by N is of course too small to be regarded in the solid arch with a continuous web. See art. 84 for its effect on braced arch. Call the distance of any fiber from nn, v; its length before flexure is $s+va$, after flexure $s+va'$. Denote the modulus of elasticity by E, and the area of the cross section of the fiber by A: the force on the fiber is,

$$f=\frac{v(a'-a)}{s+va}\mathrm{EA}$$

If r is the mean radius of curvature at the cross section, $ra=s$, which substitute above. Now the sum of the moments of the forces f, above and below nn, must equal the moment of the external forces $\mathrm{R}\overline{\mathrm{R}}=\mathrm{M}$.

$$\therefore \mathrm{M}=\Sigma\frac{v^2(a'-a)}{(r+v)a}\mathrm{EA}$$

The term $\Sigma v^2\mathrm{A}=\mathrm{I}=$moment of inertia of cross section. In taking the above

sum, we see that if the cross section is symmetrical with respect to nn, that for equal values of v above and below nn, the sum of the two moments

$$\frac{\mathrm{E}v^2\varDelta a\mathrm{A}}{(r+v)a}+\frac{\mathrm{E}v^2\varDelta a\mathrm{A}}{(r-v)a}=\frac{2}{ra}.\frac{\mathrm{E}\varDelta av^2\mathrm{A}}{1-\frac{v^2}{r^2}}.$$

When $\frac{v^2}{r^2}$ is very small, as it generally is, it may be neglected, so that regarding E and r as constant for the length s, we have

$$\mathrm{M}=\frac{\varDelta a.\mathrm{EI}}{s} \quad . \quad . \quad . \quad . \quad (1)$$

$$\therefore \quad \varDelta a=\frac{\mathrm{M}.s}{\mathrm{EI}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

10. If we denote by f' the strain per unit of area on the fiber, most distant from nn, whose distance from nn is v', we have from the value of $\frac{f}{\mathrm{A}}$,

$$f'=\frac{v'.\varDelta a.\mathrm{E}}{(r+v)a}=\frac{v'.\varDelta a.\mathrm{E}}{s},$$

when v' is small compared to r; so that (1) may be written,

$$M=\frac{f'}{v'}I \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It will be well for the reader to bear carefully in mind the approximations introduced.

We have omitted the deformation due to T, for the present. In the solution given in Du Bois' Graphical Statics, this shortening of the axis, is included from the beginning. Its influence, however, is generally very small. We shall include its effect under the head of temperature strains.

11. It may be observed that in a curved beam the neutral axis does not exactly coincide with the center of gravity of the cross section, since the elongations *per unit of length* of the fibers, is not the same at equal distances either side of the real neutral axis, though very nearly so for flat arches. Assuming nn to coincide with the centers of gravity of the successive cross sections, we find the strain on the outer most fibers due to M and force T as follows:

From the middle of s draw a perpendicular to nn to intersection with R, and call its length p. Now if R is decomposed at this intersection into components N and T, the former produces no moment about the middle of s; hence $M=Tp$.

Calling A the area of the cross section and g its radius of gyration, $I=g^2A$, and (3) may be written

$$Tp=\frac{f'}{v'}g^2A \therefore f'=\frac{T}{A}\frac{v'p}{g^2}.$$

The force T, at the middle of s, produces a uniform strain on the fibers of the cross section there $=\frac{T}{A}$, so that the total strain on the outermost fiber per unit is,

$$f_0=\frac{T}{A}\left(1\pm\frac{pv'}{g^2}\right) \quad . \quad . \quad . \quad (4)$$

12. The last two articles are only of service in estimating the maximum strain on any fiber when the position of R is known. We must now deduce principles by which to find the locus of the resultants on all the cross sections.

The angle $(a'-a)=\Delta a$ (Fig. 3) is the *change* of inclination of the tangents at the ends of the arc s due to the moment M producing flexure.

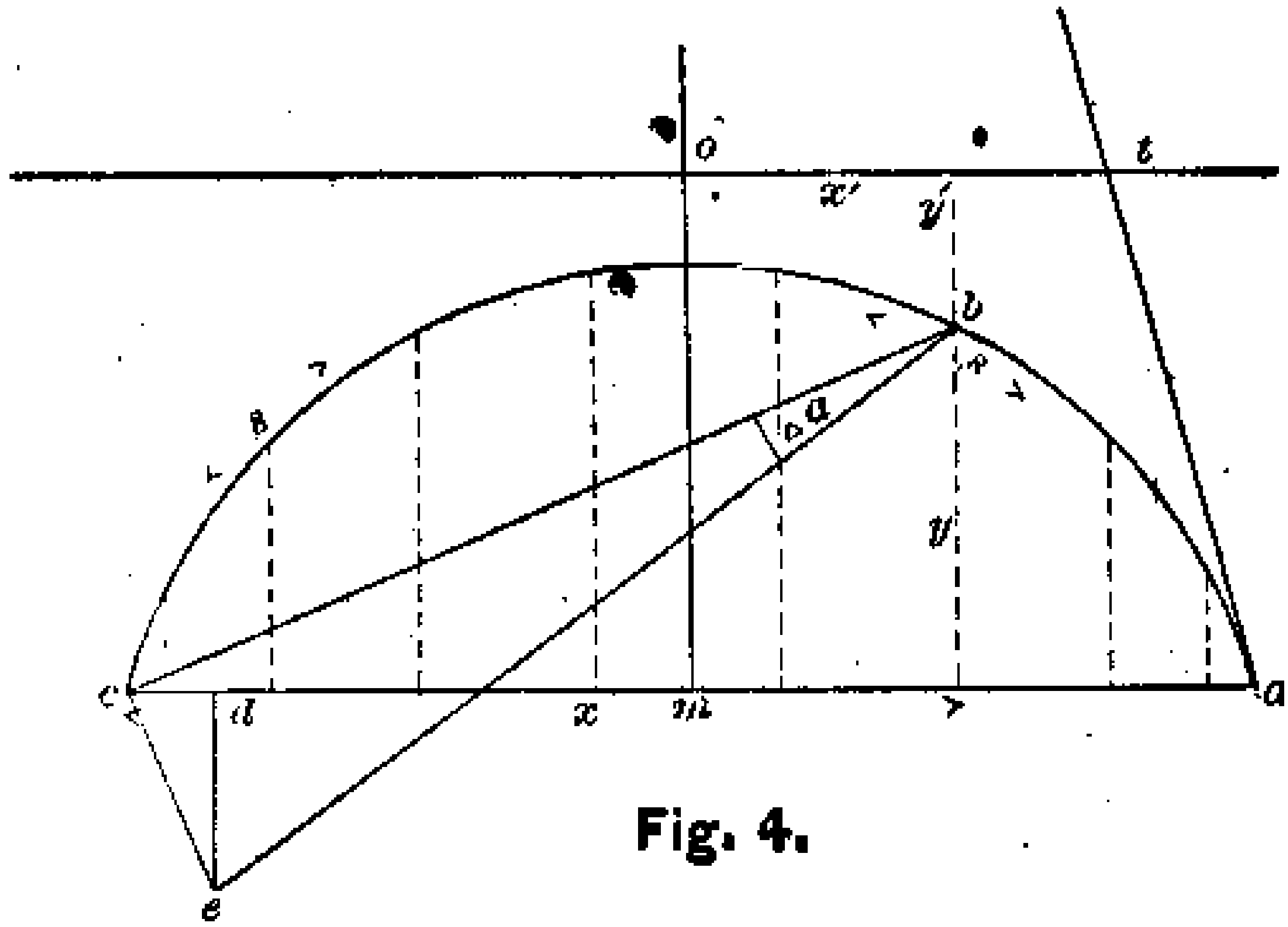


Fig. 4.

In Fig. 4 let the arc *abc* represent the neutral line *nn* of Fig. 3. Also let *b*, whose co-ordinates with *c* as the origin are x and y, be the center of a portion s of the neutral line. Let Δa for this portion = angle *cbe*. At *c* draw *ce* ⊥ *bc* and *ed* ⊥ *ac*, or the axis of x.

Then from similarity of triangles

$$\frac{cd}{ce}=\frac{y}{bc} \therefore cd=\frac{ce}{bc}y=y.\Delta a$$

$$\frac{de}{ce}=\frac{x}{bc} \therefore de=\frac{ce}{bc}x=x.\Delta a.$$

The angle Δa being very small we have replaced its tangent by the arc itself.

As before observed, the moment M, at the middle of the little arc considered, s, being small, is taken, approximately, as producing the same change Δa, as the real moments acting. Again, the change Δa is made up of the numerous little changes occurring at each little elementary portion of the arc s, so that if each portion was considered as above, x and y would have values above and below the co-ordinates of the middle point b. We assume, therefore, that if M, x and y are taken at the middle point, b of the arc s considered, that the true horizontal and vertical displacements of c are given very nearly by the above equations. Some such approximation is inseparable from the graphical method. Therefore divide up the arc *abc* into a number of parts, deduce the horizontal and vertical

displacements for each portion and find their sums, $\Sigma y.\Delta a$, $\Sigma x.\Delta a$.

If the tangent at a moves through a small angle β, we have de due to it$=x\beta$ $=\overline{ac}.\beta$; the horizontal displacement is $y\beta=o$. So that, calling h and v the total horizontal and vertical displacements of the left end of the arch, we have,

$$h=\Sigma_a^c(y\Delta a) \quad . \quad . \quad . \quad (5)$$

$$v=\Sigma_a^c(x\Delta a)+\beta.\overline{ac} \quad . \quad (6)$$

It is believed that the results attained by measuring x and y to the middle of each little elementary arc are more nearly correct, than when they are measured to the extremities of the little arcs.

The arc may be divided into equal or unequal parts. It can easily be divided into *equal* parts by the dividers: thus, dividing abc into two parts, these parts into two others and so on. The result is evidently more correct the greater the number of divisions.

13. From eq. 2, we find Δa for one

elementary arc. By adding the values found for each portion, we have for the total change of the inclination of the tangent at c with respect to the tangent at a

$$\Sigma(\varDelta a) = \Sigma_a^c \frac{\mathrm{M}s}{\mathrm{EI}} \quad . \quad . \quad . \quad (7)$$

Substituting the value of $\varDelta a$ from (2) in (5) and (6), we have,

$$h = \Sigma_a^c \frac{\mathrm{M}ys}{\mathrm{EI}} \quad . \quad . \quad . \quad (8)$$

$$v = \Sigma_a^c \frac{\mathrm{M}xs}{\mathrm{EI}} + \beta.\overline{ac} \quad . \quad . \quad (9)$$

If we divide the neutral line abc into equal parts s, as suggested, s may be placed before the sign of summation.

We must now find M, E, I, x and y, for the middle of each part and sum the separate displacements for each part between the limits a and c, as in fact, the equations indicate.

We shall generally, hereafter, for simplicity, suppose E and I constant, so that they may be placed before the sign of summation.

It may be well to state here, that if

the actual moments M, acting on each division of the arch, are known, together with E and I, that we can find from eqs. (8) and (9) the horizontal and vertical displacements of each point of the arch with reference to a, by regarding the point c, the origin of co-ordinates, at each point in turn, and effecting the summation from a to c. We can thus compute, and lay down on a drawing, the exact form assumed by the neutral line after strain. By this means we may predict the lowering of the crown or spreading at the haunches, which last in the stone arch, is generally resisted partially by the spandrels. This partial resistance, however, causes a new curve of pressures, as we shall see, thus changing the values of M, so that it does not seem possible by this means to compute the actual resistances of the spandrels.

14. From the last three equations we obtain the conditions necessary to locate the positions of the resultants of the external forces at every cross section. Thus, *for an arch whose tangents at the*

abutments are fixed in direction, span invariable and vertical deflection of end zero, i.e., for an *arch fixed at the ends*, then eqs. (7),(8) and (9), and β, are each equal to zero.

$$\therefore \quad \sum_a^c \mathrm{M}=o, \quad \sum_a^c \mathrm{M}y=o, \quad \sum_a^c \mathrm{M}x=o$$

15. *If the arch is fixed in direction at a, but hinged at c*, so that the tangent there may change, we have $h=o$, $v=o$, $\beta=o$,

$$\therefore \quad \sum_a^c \mathrm{M}y=o \quad \sum_a^c \mathrm{M}x=o$$

the origin of co-ordinates being at the hinged end.

16. *If the arch is hinged both at a and c*, the span being invariable,

$$\sum_a^c \mathrm{M}y=o$$

It is true that $v=o$ in eq. 9, but since β now has some value, it follows that $\Sigma \mathrm{M}x$ cannot be zero as before.

17. *Lastly, if the arch is fixed in direction at the ends, and hinged at some intermediate point b*, the horizontal and

vertical displacements of b, for the part ab, must be the same as for the part bc, owing to the connection. So that we have the conditions,

$$\Sigma_a^b \mathrm{M}x = \Sigma_c^b \mathrm{M}x, \quad \Sigma_a^b \mathrm{M}y = -\Sigma_c^b \mathrm{M}y.$$

The last term is minus, since if $\Sigma\mathrm{M}y$ on one side is plus, on the other side it is minus, to cause the horizontal displacement for the two parts to be in opposite directions. From the reasoning of art. 12, we see that the origin of co-ordinates for this case must be taken at b, y vertical, x horizontal as before.

18. For the arch "fixed at the ends" we may take the origin of co-ordinates at some other point than c, as o, if preferred.

For we have, calling $\overline{cm}=d$, and $\overline{om}=d'$, the old co-ordinates of b, x and y, the new, x' and y': $x=d-x'$, $y=d'-y'$, so that (art. 14)

$$\Sigma\mathrm{M}x=\Sigma\mathrm{M}(d-x')=d\Sigma\mathrm{M}-\Sigma\mathrm{M}x'=o$$
$$\Sigma\mathrm{M}y=d'\Sigma\mathrm{M}-\Sigma\mathrm{M}y'=o$$
$$\therefore\ \Sigma\mathrm{M}x'=o,\ \Sigma\mathrm{M}y'=o,$$

since by the first condition $\Sigma M = o$. It is important to observe that x' must change sign on the other side of om in the summation. Similarly for y' if o is below the crown.

If we are using a trial curve of pressures for which ΣM is not zero, we must of course reckon the origin at c to find the real sums, which may not be zero if the curve is not located properly.

Similarly for an arch, with one or more hinges, the origin must be at the hinged point, since ΣM is not zero, &c.

19. If in any case the arch is not hinged on the neutral line abc, but at some point outside of it, this point must be taken as the origin of co-ordinates, since for this point the condition that the horizontal and vertical displacements equal zero is alone true, as follows from art. 12. If the ends of the arch are flat against the skewbacks, but not bolted to them, the bending moment at the abutment may become large enough to lift one edge clear of the abutment, which presents a case similar to the above. It

is plain that we divide up the *neutral line* *abc* into equal parts as before, and that we must now attend to the signs of x and y, if any of the values are minus. This case has been treated by Mr. Charles Pfeifer, C.E. (V. N.'s Mag. for June, 1876), though the use of his eqs. (17) and (18) involve the principle that the deflections of the end of the centre line are zero; whereas the true conditions are that the deflections of the hinged, or free-to-turn point are zero. The other view would involve our taking x and y about c in Fig. 4, in place of the point about which the arch rotates. This error though is generally very small. In DuBois' *Graphical Statics*, p. 388, art. 22, is given an example of "Arch continuous at crown and hinged at ends of lower rib," the formulae deduced for arches hinged in the neutral line at the ends being supposed to apply, which is evidently incorrect.

20. When the moment of inertia and modulus of elasticity vary for different parts of the arch, we have only to replace

M in arts. 14 *et seq.* by $\frac{M}{EI}$, when the above conditions hold. In designing an arch, it is of course impossible to know the values of I beforehand, so that some section must be assumed by which to compute a more correct one, and this in turn assumed, and so on; by this means approximating to the true result.

PRESSURE CURVE.

21. In Fig. 5, let the force P be in equilibrium with the forces V_1 and F at c_1 and V_2 and F at c_2, the forces F being parallel. The resultant of V_1 and F at c_1 must intersect the resultant of V_2 an F at c_2, in some point c_3 of P. Lay off P on the load line on the left, draw R_1 and R_2 parallel respectively to c_1c_3 and c_2c_3 to intersection o. Then R_1=resultant of V_1 and F at c_1, and R_2=resultant of V_2 and F at c_2. Hence, drawing the line F through o, parallel to force F, we see that the forces at c_1 and c_2 must have the intensities shown by the sides of the triangles of the force diagram on the left.

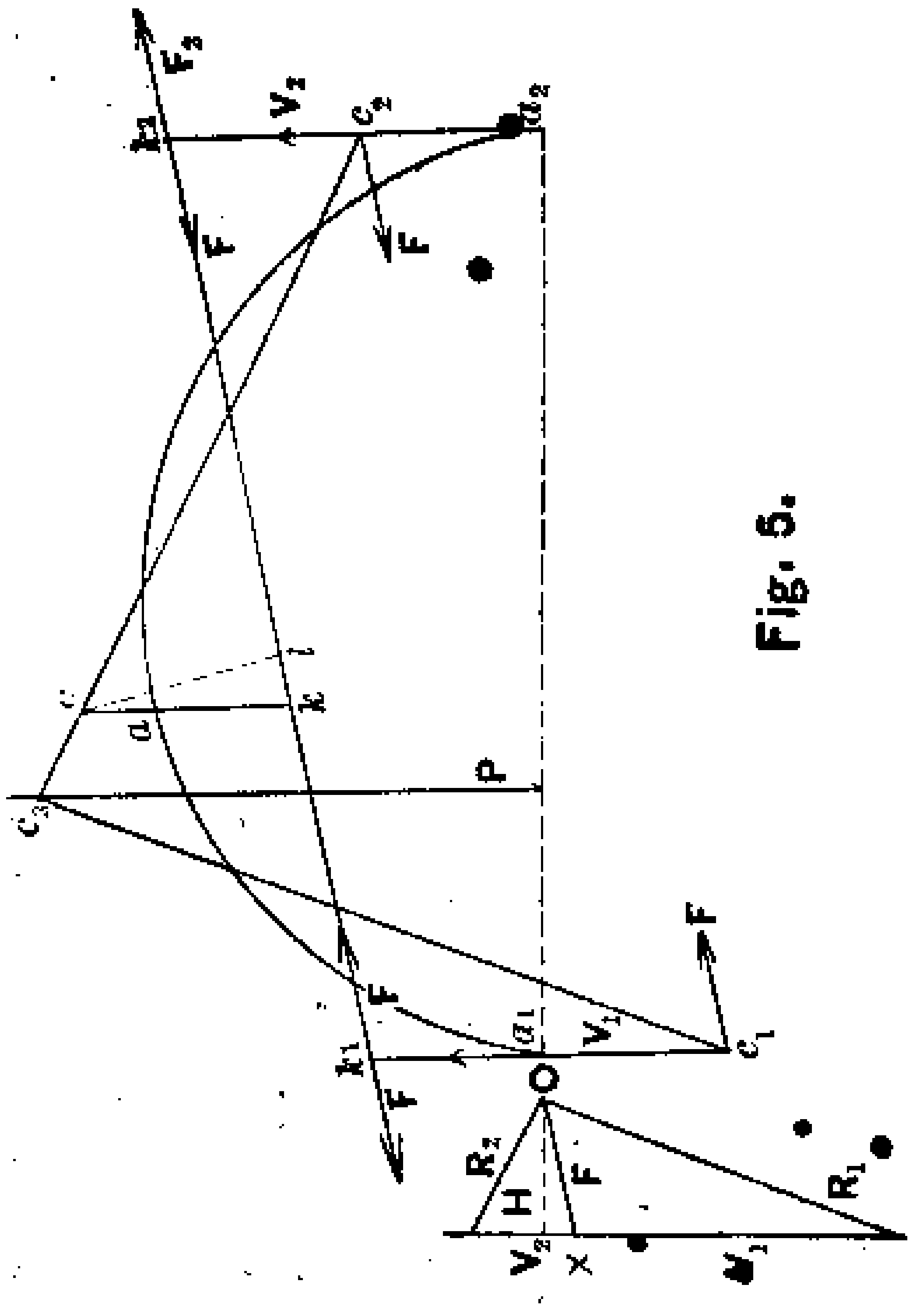


Fig. 5.

Now draw a line k_1k_2 parallel to the direction of force F, also draw the lines c_1k_1 and c_2k_2, parallel to P. It will not affect equilibrium to add two equal opposed forces F at k_1, also at k_2 acting parallel to F, all of them; but since the two forces that act towards each other now balance, the effect is to form a couple at the left with the F at c_1 and the remaining F at k_1, similarly at the right. If we denote the angle made by F with a line perpendicular to P by a, we have the moment of the couple at $c_1 = F. \overline{k_1c_1}. \cos a = H. \overline{k_1c_1}$ and at c_2, $F.\overline{k_2c_2}. \cos a = H.\overline{k_2c_2}$, H denoting the pole-distance of force polygon, as shown by the dotted line from o.

Now since our system of forces is balanced at each apex, k_1, c_1, c_3, c_2, k_2, on supposing opposed forces R_1, R_1 acting along c_1c_3 and R_2, R_2 along c_3c_2, as we see by reference to the force polygon, if we suppose the forces removed to the right of the line ck, eqilibrium still holds at c_3, c_1 and k_1, so that the sum of the moments of the forces there about

any point c must be zero. Suppose the two opposed forces R_1, acting along $c_1\, c_3$, removed, we have left P, V_1, couple $\widehat{FF}$, force F acting to the right at k_1, and force R_2 on cc_3. The moment of this last is zero; hence, denoting the lever arms of V_1 and P about c by x and d, we have

$$(V_1 x - H.\overline{c_1 k_1} - Pd) = F.\overline{cl},$$

$\overline{cl}$ being perpendicular to $k_1 k_2$. Draw kc parallel to P; then from similarity of a triangle in force diagram to ckl, we have $F.\overline{cl} = H.\overline{kc}$. Now if we had a structure, as a continuous girder acted on by V_1, couple FF and P to the left, we should call the term $(V_1 x - H.\overline{c_1 k_1} - Pd) = M_c$, the moment of the external forces left of the section taken; whence,

$$M_c = H.\overline{kc}$$

This same result holds to the left of P, the term Pd being then omitted. If c is taken below the closing line $k_1 k_2$, as we shall call it, the moment of F is of an opposite kind to the first, so that M_c changes sign.

We see from the foregoing that $k_1\, c_1\, c_2\, c_5\, k_2$ is the equilibrium polygon for the forces considered, and that the moment M_c at any point is equal to the pole distance H multiplied by the ordinate $\overline{kc}$ of the equilibrium polygon to the closing line. The effect of an end moment is thus to shift the closing line at the end an amount equal to the moment divided by H. If the forces of the end couple as given, do not coincide with F and F, we can replace it by another couple of same moment whose forces do coincide with F and F as drawn.

If there are a number of forces parallel to P, the equilibrium polygon c is formed as usual, and the above relation evidently holds.

22. Let us now conceive $a_1 a a_2$ (Fig. 5) to be the neutral line of an arch rib fixed at the ends and acted on by a single weight P, the real reactions being R_1 and R_2, acting at c_1 and c_2, respectively, in the directions $c_1 c_3$ and $c_2 c_3$. If we decompose each reaction into horizontal

and vertical components, we see that the former alone produce the end moments which cause fixity; these moments being $H.\overline{a_1c_1}$ and $H.\overline{a_2c_2}$ respectively.

In case that the arch is hinged at the end, there is no moment, so that the reaction must then pass through a_1, a_2, or both, according as the arch is hinged simply at a_1 or at a_2, or at both of these points. Now decompose R_1 and R_2 at c_1 and c_2 into components V_1, F and V_2, F.

As before, let us apply along some line k_1k_2, parallel to the direction of F, four forces F acting away from each other at k_1 and k_2 as shown in the figure. This does not disturb equilibrium, but it has the effect of transferring F at c_1 and c_2 to k_1 and k_2, and of adding the couples $H.\overline{k_1c_1}$ and $H.\overline{k_2c_2}$ at the left and right abutments respectively.

Taking moments of the forces to left of a about a we have,

$$M=(Vx-H.\overline{k_1c_1}-Pd)-H.\overline{ak};$$

noting that the moment of F at k_1 about $a=H.\overline{ak}$. Now we see that the term

$(Vx - H.\overline{k_1c_1} - Pd)$ *is the moment at a of a girder acted upon by the force* P, *with end moments*, $H.\overline{k_1c_1}$ and $H.\overline{k_2c_2}$ *at* a_1 *and* a_2 *respectively, the pole distance* H *being the same as the actual horizontal thrust of the arch; and that* $k_1c_1c_3c_2k_2$ *is the equilibrium polygon for such a girder, &c., so that the term above,* $M_c = H.\overline{kc}$. Hence we find the bending moment at any point *a* of the arch to be,

$$M = H(\overline{ck} - \overline{ak}) = H.\overline{ac}.$$

Hence *the actual moment at any point of an arch is equal to the horizontal thrust multiplied by the vertical distance from the equilibrium polygon c to the neutral line of the arch,* M *being + or –, as* $\overline{ck} > \overline{ak}$, *or* $< ak$. This curve $c_1c_3c_2$ is the locus of the real resultants acting at points of the arch, and is called *the pressure curve.* It is evidently irrespective of the particular position of k_1k_2, and hence of the end moments $H.\overline{k_1c_1}$ $H.\overline{k_2c_2}$.

Remark.— This result is easily proved otherwise. In fig. 3, from the middle point of arc *s*, draw vertical and horizontal lines to intersection with R and denote their lengths by *v* and *h*; also

draw a perpendicular line to R, and call its length p. If the hypothenuse of the large right triangle formed represents R, the side h will represent H. From similarity of triangles we have.

$$\frac{R}{H}=\frac{v}{p} \quad \therefore \quad Rp=Hv, \text{ as stated,}$$

23. Now if we regard the neutral line of the arch itself as an equilibrium polygon, with a closing line k_1k_2, due to some kind of loading not given, having a pole distance H equal to the actual horizontal thrust of the arch, the term $H.\overline{ak}$ in the previous equation is the moment of the supposed external forces at a and may be denoted by M_a, so that the previous equation may be written,

$$\mathbf{M}=\mathbf{M}_c-\mathbf{M}_a \quad . \quad . \quad . \quad . \quad (10)$$

This equation holds on the supposition that c_1 and c_2 are the real points of action of the resultants at left and right abutments; but there are no restrictions as to the position of k_1k_2 whatsoever. We shall presently see the necessity of imposing some. Thus for the arch *fixed at the ends*, we have (art. 14)

$$\Sigma M = o,\ \Sigma Mx = o,\ \Sigma My = o \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Now if we suppose the end moments, $H.\overline{k_1 c_1}$ and $H.\overline{k_2 c_2}$ of sufficient intensity to fix in direction the tangents to the neutral line of a girder, acted upon by the same loading as the arch; and if the vertical deflection of one end above the other is zero, we have, since M_c is the moment for such a girder,

$$\Sigma M_c = o \ \Sigma M_c x = o \quad . \quad . \quad . \quad . \quad (12)$$

If we subtract the first two of eqs. (11) from eqs. (12), we have as a necessary consequence from eq. (10),

$$\Sigma M_a = o \ \Sigma M_a x = o \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Now we shall find that by eqs. (12), we are enabled to find the end moments $H.\overline{k_1 c_1}$, $H.\overline{k_2 c_2}$, without knowing H, or the position of the closing line. We shall further find that eqs. (13) enable us to find this closing line, $k_1 k_2$, in position. Lastly from the condition,

$$\Sigma My = \Sigma(M_c - M_a)y = o,$$

we will be enabled to find H, whence from the found end moments, the dis-

tances k_1c_1, k_2c_2, may be ascertained and laid off from k_1 and k_2, thus giving us two points of the pressure curve, whence it may be drawn, with the known value of H.

24. It is true that we may determine some other closing line by assuming,

$$\Sigma M_c = A, \ \Sigma M_c x = B$$

whence by subtraction, as before, we have as a consequence,

$$\Sigma M_a = A, \ \Sigma M_a x = B$$

But it is apparent since the total bending is now proportional to A (see art. 13) that the vertical deflection is no longer zero, but dependent on the value assumed for A, so that B is a function of A and cannot be assumed arbitrarily. But as this involves a computation of B, the steps are not near so simple, as when we assume the total bending of the supposed girder zero, when of course the vertical deflection is so likewise.

25. From similar considerations of

simplicity, it will be assumed that when an arch is hinged at any point or points, that the supposed girder is similarly hinged, so that the closing line k_1k_2 will pass through the point or points (not exceeding two) since the moment there is zero.

The solution corresponding to Fig. 5 comprehends every case, the closing line changing its position according to the conditions of the case.

Thus for the cases treated in arts. 14–17, the formulae there given for the *arch* hold.

It is then evident that whichever of the similar conditions apply to the *girder*, apply likewise to *curve a* regarded as an equilibrium polygon as follows from the relations expressed by eq. 10.

26. We have hitherto, for simplicity, considered but one weight P on the arch; but since for any number of weights the two straight lines c_1c' and $c'c_2$ are simply changed into a polygon cc, the equilibrium polygon (art. 2) due to the weights alone, the result holds for any loading

whatsoever. The above demonstration is essentially the same, with some emendations, as that given by the writer in VAN NOSTRAND'S MAGAZINE for June, 1878.

27. The above principle was first stated by Prof. H. T. Eddy in the January, 1877, No. of VAN NOSTRAND'S MAGAZINE, in these words:

"If in any arch that equilibrium polygon (due to the weights) be constructed which has the same horizontal thrust as the arch actually exerts; and if its closing line be drawn from consideration of the conditions imposed by the supports, etc.; and if, furthermore, the curve of the arch itself be regarded as another equilibrium polygon due to some system of loading not given, and its closing line be also found from the same considerations respecting supports, etc.; then when these two polygons are placed so that these closing lines coincide and their areas partially cover each other, the ordinates intercepted between these two polygons are proportional to the real bending moments acting in the arch."

As Prof. Eddy claims, this proposition supplies the hitherto missing link in the graphical treatment of solid or braced arches.

In art. 34, see an equally general method, which does not involve a demonstration of this proposition, though identical with it in the construction.

ARCH WITH FIXED ENDS.

28. Let the curve aa_4a (Fig. 6) which may be of any kind, be the neutral line of an arch rib. Divide it into equal parts (8 in the figure), and at the middle of these parts $a_1, a_2, \ldots$ draw vertical ordinates y to the axis of x, aa. The origin may be taken at either end of the span in this case.

Conceive the vertical loads to be applied at the points $a_1, a_2, \ldots$, of the intensities $P_1, P_2, \ldots$

Lay off these loads to scale on the load line P_1P_8, choose a pole o, and beginning at some point v in the vertical through a draw $vp_8 \parallel$ ray VP_8 (it coincides with it in the Fig.) then $p_8\, p_7 \parallel$ ray

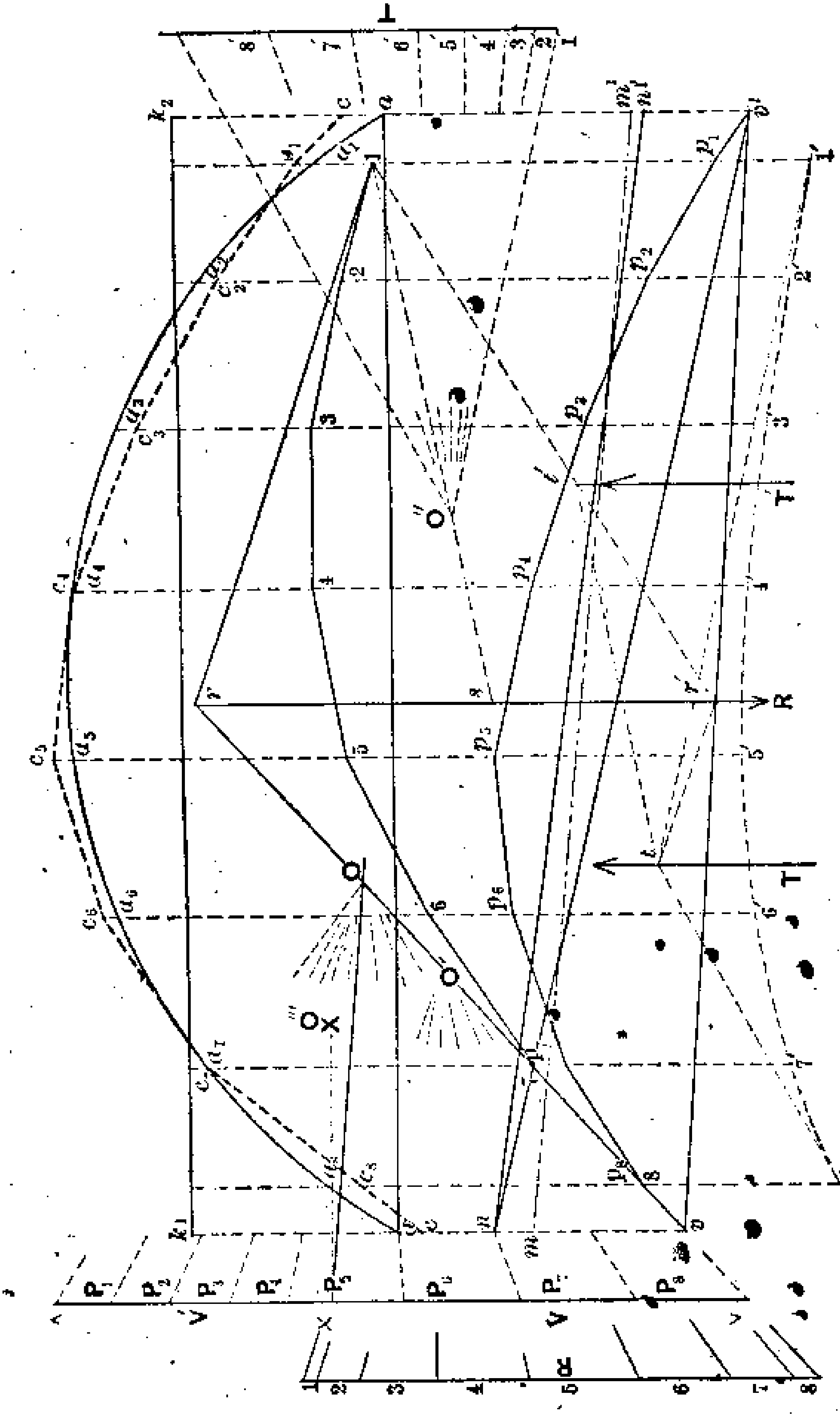


Fig. 6.

P_7P_8, etc., to intersections with the verticals through a_8, a_7, etc. From the last point v', so found, draw a closing line vv'; a line through o parallel to it would give the reactions of the arch acting as a girder with no end moments, since vp_5v' is the equilibrium polygon for the forces (art. 2). It will conduce to brevity to designate the curves or polygons that are marked with the same letter, though different subscripts, by that letter; thus, the curve $aa_1a_2 \ldots a_8a$ will be called the curve a; similarly we have the polygons p and c.

In order to find the true closing line for the arch acting as a girder, we have the conditions, art. 23,

$$\Sigma M_c = o, \quad \Sigma M_c x = o.$$

Let nn' be a trial closing line, then since the bending moments M_c are proportional to the ordinates of the equilibrium polygon, $nvpv'n'$, the first condition requires, that the sum of the vertical ordinates through $p_1, \ldots, p_8$, from nn' downwards to pp = sum of ordinates

above nn' to polygon pp. It is easily seen from the considerations of art. 4, that the bending moments are of different sign when the ordinates lie on opposite sides of the closing line.

If we add to the above equality, the sum of the ordinates intercepted between vv', the line nn', and the part of the polygon p below nn, we have the condition that the sum of the ordinates included between vv' and nn' shall equal the sum of the ordinates from vv' to polygon pp; hence if we denote the ordinates from vv' to pp and nn' respectively by p and n, and the ordinates between nn' and the curve pp by m, we have,

$$m=p-n$$

The second condition, $\Sigma M_c x=o$, may be written $\Sigma mx=\Sigma px-\Sigma nx=o$; which indicates that if the ordinates p and n are regarded as forces, that the sum of their moments about an abutment are equal; *i.e.*, the resultant of the p's coincides with the resultant of the n's, since

$\Sigma p = \Sigma n$, from the conditions $\Sigma m = o$, as proved above.

Therefore find the resultant R of the ordinates p regarded as forces, in position, direction and magnitude. This may be done by calculation, or the graphical method given in art. 3 may be more conveniently employed. In the latter case we have only to lay off the ordinates p ($\frac{1}{2}p$ was taken as more convenient) on a vertical line 18 precisely as we did P_1, P_2, ..., choose a pole o', then starting at p_8 say, draw the equilibrium polygon 87654321 as usual. On prolonging the first and last sides, corresponding to vp_8 and $v'p_1$ of the first polygon, to intersection r, a vertical through this point will give the position and direction of R. In magnitude, $R = \Sigma p$, as taken from the force line.

29. Now draw a line nv' dividing the ordinates n into two sets. Lay off the half of the ordinates (through each point of division a_1, a_2, . . .) intercepted between vv' and nv', as forces on the line 8'...1', and similarly find their resultant

T in position. Now the position of T is not changed when nn' is changed, by revolving nv' about v' through some angle: for all of the ordinates in the triangular portion $vv'n$ are altered in the same ratio, as is shown in *Geometry* (see Chauvenet's Bk. III, Prop. 8).

It follows that T′, the resultant of the half ordinates included between nv' and nn', is at the same distance from v' that T is from v.

The problem then is this, having R in position, &c., and T and T′ in position only, what are the real magnitudes of T and T′?

Laying off $\overline{rr'}=$R, draw from some point 1 $\overline{1r}$ and $\overline{1r'}$. Next through some point r' on R draw $\overline{r't} \parallel 1r$ and $r't' \parallel 1r'$ to t and t'; a line $\overline{1s}$, drawn parallel to the closing line tt', divides $\overline{rr'}$ into segments rs and sr' equal to T and T′ respectively. Now if the load line $\overline{8'1'}$, which represents the trial T, is not equal to $\overline{rs}$, we must reduce the length vn to vm in the ratio of $\overline{8'1'}$ to $\overline{rs}$, for then each ordinate in the triangle $vv'n$ is reduced in this ratio, so

that their half sum will now equal $\overline{rs}$. Similarly the half sum of the ordinates in the triangle $nv'n'$ (which can be taken by dividers) should equal $\overline{r's}$, if n' has been accurately located. If not change $\overline{v'n'}$ to $\overline{v'm'}$ in ratio of $\overline{r's}$ to the half sum mentioned, when all the ordinates in triangle $nv'n'$ will be altered in this ratio, and hence their sum. Now the sum of similar ordinates in the triangle $nv'm'$, is the same as for the triangle $mv'm'$, the lengths being the same in either case; whence mm' is the true closing line to satisfy the two conditions $\Sigma M_c = o$ and $\Sigma M_c x = o$. It is well to test the first, by adding the ordinates to mm' from curve p, and see whether the sum of those above mm' is equal to the sum of those below the same line.

30. It is well also to choose the poles o' and o'' so that the exterior rays will form angles of nearly 90° with each other, as then the positions of r and t can be determined more accurately. The ordinates were halved for the force lines simply, to economize space. When the

loading is symmetrical with respect to the crown (as for a uniform load, for example) the ordinates of the polygon *p* are symmetrical about the center vertical, hence the closing line mm' is drawn parallel to vv', and is quickly fixed by the condition $\Sigma M_c = o$ as before. The other condition $\Sigma M_c x = o$ holds also; in fact the requirement that mm' be parallel to vv' is a consequence of this condition.

31. Let us next fix the closing line k_1k_2 of the curve *a*, regarded as an equilibrium polygon, by the conditions, $\Sigma M_a = o$, $\Sigma M_a x = o$.

The resultant of the ordinates from the span line *aa* to curve *a* passes through the crown in consequence of the symmetry of curve *a* about the crown, also the resultant of the ordinates from span *aa* to k_1k_2 (see art. 28), since then the condition $\Sigma M_a x = o$ is satisfied. But the latter resultant cannot pass through the crown unless k_1k_2 is horizontal. Therefore draw a trial *kk*; add, by the dividers, the ordinates below it to curve *a*, also those above it; their difference

(if any) divided by the number of ordinates (8 in this case) will give the amount to raise or lower $\overline{kk}$ so that the conditions $\Sigma M_a = o$, is satisfied.

32. We have now the condition

$$\Sigma My = \Sigma(M_c - M_a)y = o$$

as the last one by which to locate the curve c in its true position, and determine the true horizontal thrust of the arch.

From the last equation we have

$$\Sigma(M_c y) = \Sigma(M_a y)$$

Now M_c is proportional to the ordinates of curve p from its closing line mm'; those above it being regarded as plus, say, those below it minus. Similarly M_a is proportional to the ordinates from $\overline{kk}$ to curve a; both curves being supposed to have the same pole-distance. Hence we sum the above products, and if the equality does not hold, we must change the pole-distance of curve p (which was assumed arbitrarily) until it does subsist. We can get the above sums, and their ratio, by calculation, by scaling off

the ordinates, multiplying y by its corresponding M, &c., but the graphical method is here very direct and even accurate.

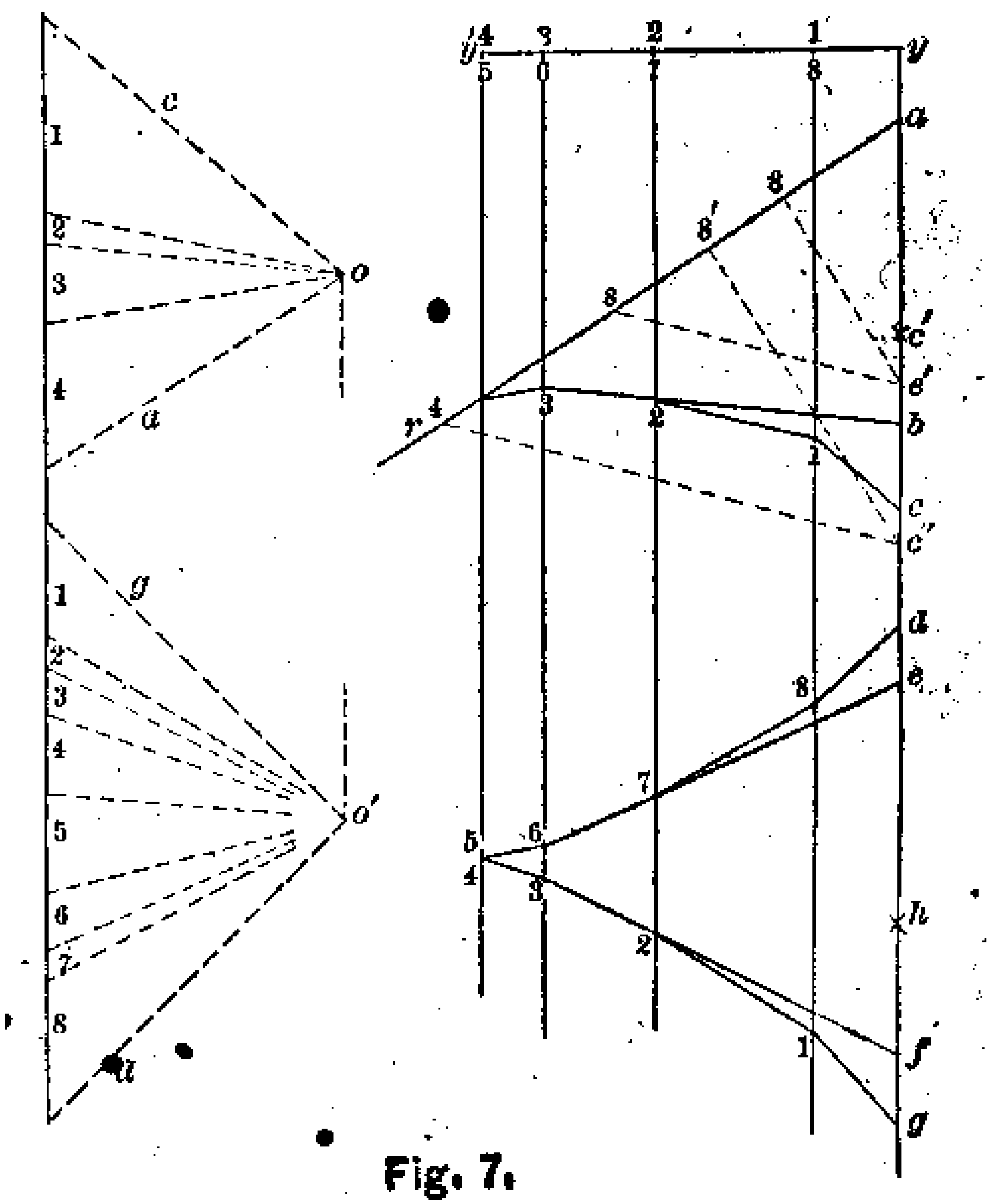
To employ it, for the object in view, draw a line yy' (Fig. 7) $\perp yg$, and lay off from y the distances, $y1$, $y2$, etc., equal to the ordinates y at a_1, a_2, etc., and draw lines $\parallel yg$ through 1, 2, etc.

Next on a line $\parallel yg$ lay off in order the ordinates from kk to curve a. We have only laid off the ordinates at a_1, a_2, a_3, a_4, since the sum $\Sigma M_a y$ is the same on the other side of the crown. Then choosing a pole o, and starting at some point c on yg, draw $c1 \parallel$ ray $c1$, $12 \parallel$ ray 12, . . . $4a$ ray $4a$ to intersection a with yg; also produce 32 to intersection b with yg; then art. 7,

$$\Sigma M_a y = 2H'(ab - bc) = 2H'.\overline{ac'},$$

on making $bc' = bc$.

Again, lay off on a line $\parallel yg$, in order, the ordinates proportional to M_c, intercepted between mm' and polygon p. With the same pole distance H′ choose a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c''
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f
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Fig. 7.

pole o'; then starting at g on yg draw $g1 \parallel$ ray $g1$, etc., and produce the sides 32 and 67 to f and e, respectively. Then we have,

$$\Sigma M_c y = H'[ef-(de+fg)] = H'.\overline{eh},$$

if $\overline{fh} = de + fg$, since $H'.\overline{ef}$, by art. 7, is the sum of the products $M_c y$ for the ordinates above kk, and $H'(de+fg)$; a similar sum for those below. Hence laying off $\overline{ac''} = 2\overline{ac'}$ and $\overline{ae'} = \overline{eh}$, since they are not equal, we must diminish the pole distance of polygon p in the ratio $\frac{ae'}{ac''}$, which thus has the effect of increasing the ordinates from mm' to polygon p in ratio $\frac{ac''}{ae'}$, in order that the equality above may hold. Thus, to find the true pole distance, we lay off the assumed pole distance from o to the line P_1P_8 of Fig. 6 on some line as $\overline{ar}$ Fig. 7 from a to r. On drawing $e's \parallel c''r$, the distance $\overline{as}$ is the true pole distance. Hence draw a horizontal line, through the point of intersection of the line drawn through

$o \parallel mm'$ with P_1P_8 (Fig. 6), a distance to o''' equal to as (Fig. 7). The reactions V and V′ are the same as before, and since our actual closing line k_1k_2 in position is horizontal, the left figure is the true force polygon corresponding to the $M_c s$. Now by art. 4, we have to elongate the ordinates from polygon p to mm in ratio $\frac{ac''}{ae'}$. Thus lay off the ordinate through p_8 (Fig. 6), from a to 8 (Fig. 7), draw $c''8' \parallel e'8$ whence $a8'$ is the true length of this ordinate. Lay it off from line k on 8th ordinate downwards to c_8.

Similarly for the other ordinates of polygon p, so that the polygon $cc_1c_2 \ldots c_8c$ may be located; or, if preferred, having found one point as c_5 in this manner, polygon c may be drawn by means of the force polygon $O'''-P$.

The extremities c and c thus found, as well as other points, should agree with those found by the first method.

33. Now it is evident that the polygon c of Fig. 6, is identical in meaning with polygon c of Fig. 5, as well as the

closing line; so that calling the pole-distance from O‴, H, we have the bending moment at any point a_5 of the arch $= \mathrm{H}.\overline{a_5c_5}$, etc.

This is plain, because polygon c of Fig. 6 satisfies the conditions $\Sigma\mathrm{M}_c = o$, $\Sigma\mathrm{M}_c x = o$, and polygon a the conditions $\Sigma\mathrm{M}_a = o$, $\Sigma\mathrm{M}_a x = o$; whence by subtraction in connection with eq. 10 (art. 22), we have $\Sigma\mathrm{M} = o$ and $\Sigma\mathrm{M}x = o$.

Again from the last condition above, we have, $\Sigma\mathrm{M}_c y - \Sigma\mathrm{M}_a y = \Sigma\mathrm{M}y = o$. So that all the conditions given in art. 23, of an arch fixed at the ends, are fulfilled.

It is well to test the ordinates of the type ac, to see if the three conditions are fulfilled. Thus the sum of the $\overline{ac}$'s above curve a, should equal the sum of those below. The sum of the products $\mathrm{M}x$ or $\overline{ac}.x$ above and below curve a should be equal; similarly for $\overline{ac}.y$, &c. The multiplications can be easily effected by the method of art. 32.

Now cc is the true *pressure curve* of the arch; *i.e.*, the reaction at the left abutment is at c, and equals the lower

ray (VP_8) through O''', its componenents being V and H; similarly the resultant ($=$ray $V'P_1$ from O''') at right abutment acts at c, its components being V' and H; so that the moments at these points are, $H.\overline{ac}$, the distance ac being different at the two abutments for an unsymmetrical load. Moreover we see that the curve c is found by combining the resultant at c, on the left, say with the force P_8, acting through the vertical at a_8; this resultant with the force at a_7, etc.; so that the rays of the force polygon $O'''P$ represent the resultants in magnitude at the middle (nearly) of the corresponding divisions. Thus, ray $O'''-P_5P_6$, is the magnitude of the resultant, acting along c_5c_6, for the middle of arc a_5a_6, &c.

On decomposing this resultant into components, parallel and perpendicular to chord a_5a_6, we have the tangential and normal components T and N of the thrust acting at, or very near, the middle of the division a_5a_6; whence by art. 11, the greatest stress on any fiber may be found. Similarly for the other divisions

of the arc. It may be observed that in place of (Tp) in eq. 4 (art. 11) we may substitute the corresponding $(H.\overline{ac})$ the $\overline{ac}$ being taken at the middle of the portion of the arc considered. We take the "middle," if the loading is continuous, for the resultants given pertain to that point of the arc where the load is supposed divided, as we see by analogy to the previous treatment of the Voussoir arch.

34. We shall now give a second general method of drawing the pressure curve c, fig. 6, that involves only the propositions of Graphical Statics given in arts. 2 3, 4 and 5 of this paper, with the remark of art. 22. Let c fig. 6, be the actual pressure curve, drawn with the force polygon $O''P_1P_8$; thus satisfying the conditions, $\Sigma M=o$, $\Sigma Mx=o$, $\Sigma My=o$, for the arch fixed at the ends. Now if we draw a trial pressure curve p (which may be supposed to occupy some such position about the arch as curve c, though it is drawn below for the sake of clearness of diagram) with an assumed pole O, we know that if

this pole is afterwards changed to O‴, that the ordinates of polygon p are altered in the ratio of old to new pole distance. This alteration, we shall see, is determined entirely from the condition $\Sigma My=o$. Now M being proportional to the ordinates of the type $\overline{ac}$ (art. 22, remark) if we denote the ordinates from some line as k_1k_2 to curves a and c by $\overline{ka}$ and $\overline{kc}$ respectively, we have $\overline{ac}=\overline{kc}-\overline{ka}$.

It is seen that $\overline{kc}$ and $\overline{ka}$ are minus when laid off below kk, since then $\overline{ac}$ is plus, when c is above a, from the preceding equation, and minus otherwise; which must be so, the moments having different signs when c is on opposite sides of curve a.

We have therefore $\Sigma\overline{ac}=\Sigma(\overline{kc}-\overline{ka})=o$ and $\Sigma(\overline{ac}.x)=\Sigma(kc-ka)x=o$. Now if for simplicity we draw kk, as in art. 31, so that $\Sigma\overline{ka}=o$, and $\Sigma(\overline{ka}.x)=o$, it follows that $\Sigma\overline{kc}=o$, $\Sigma(\overline{kc}.x)=o$, also. Now polygon p is simply polygon c out of position and with ordinates all altered in the same ratio; so that we must determine a line mm' that will satisfy the con-

ditions, $\Sigma\overline{mp}=o$, $\Sigma(\overline{mp.x})=o$, which is readily done, as shown in arts. 28 and 29. As explained above, ordinates on opposite sides of mm' or kk have opposite signs.

Lastly from $\Sigma My=o$, we have $\Sigma(\overline{ac.y})=\Sigma(kc-ka)y=o$.

$$\therefore\ \Sigma(\overline{kc}.y)=\Sigma(\overline{ka}.y) \text{ or } \Sigma(\overline{mp}.y)=\Sigma(\overline{ka}.y)$$

from which condition curve c is drawn as explained in art. 32, the alteration of ordinates not affecting the conditions $\Sigma M=o$, $\Sigma Mx=o$.

On drawing rays P_1P_2, P_7P_8 parallel to c_1c_2, c_7c_8, thus found, we find O''', the new pole, or better, the old rays, P_1P_2 can have their inclinations altered in the same ratio as the lines p_1p_2, etc. This very general method applies to arches with hinges also, as well as the fixed arch.

APPLICATION TO VOUSSOIR ARCHES.

35. It has been previously observed (art. 1) that, in certain cases, the treatment of the voussoir arch falls under that

pertaining to the solid arch. In fact, conceive a *solid* stone arch in which the pressure curve keeps within such limits that only the resistance to compression of the stone is brought into play. Now, if this solid arch is divided up into any number of voussoirs, the joints being perfectly cut, the conditions are unchanged; since the change of direction, Δa of tangents, at ends of a portion of the arch whose length is s, is the same in either case, this change being due entirely to the *compression*, on any supposed joint, not being uniform, but uniformly increasing in going towards one edge. The values $\overline{cd}=y.\Delta a$, $de=x.\Delta a$, of art. 12, are thus the same in either case, as well as the expressions for the total horizontal and vertical displacements following.

It should be noted, however, that in the value for Δa, given by eq. 2, art. 9, that the uniform thrusts T, on the cross sections, which shorten the entire arch, are neglected. Its effect on the bending is very slight, and is included, further on,

under temperature strains, and we may assume, that for the voussoir arch, eq. 2, represents the true value of Δa, and that I has the same value as for the solid arch. It must not be supposed by the last hypothesis that the voussoir arch can supply tensile resistances; we only use eq. 2, with the value of I as for a solid arch, as a very near approximation to the exact value of Δa in the case when no tensile forces are needed.

The exact expression for Δa, including the effect of the T's in shortening the axis is given in eq. (9), page 274 of Du Bois' *Graphical Statics*. It is seen from the following eqs. (12) and (13), giving values corresponding to our $\Sigma \Delta a$, how slight is the influence of this shortening of the axis.

This value of Δa having been assumed, the resulting equations and conditions for the arch fixed at the ends are the same as for the solid arch; so that the treatment pertaining to the case just examined applies.

36. Let us consider the segmental arch of 75 feet span, and 17 feet rise, given in art. 48 of *Theory of Voussoir Arches, &c.*

In Fig. 8, is shown a slightly exaggerated drawing of the arch with its load of 3,000 lbs. per foot, and the position of the pressure curve (as taken from a large drawing), corresponding to the maximum and minimum thrust in the limits of the middle third very nearly.

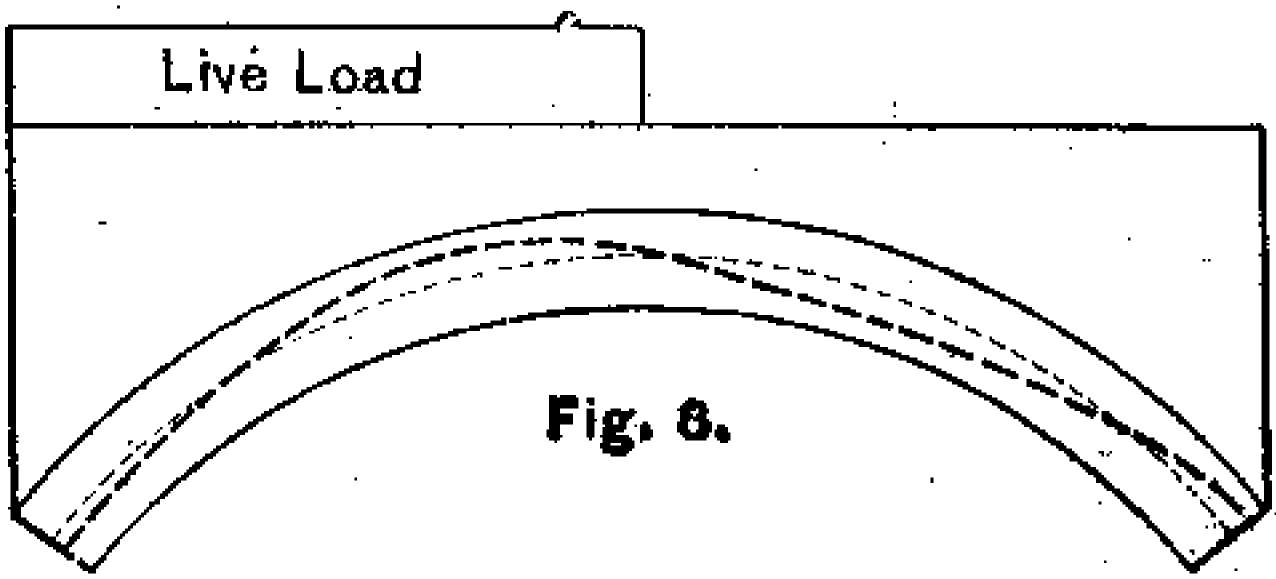


Fig. 8.

The above curve then corresponds to the trial curve p of Fig. 6.

If the greatest accuracy attainable by the graphical method is desired, the curves p and c (Fig. 6) should be drawn exactly as detailed in *Voussoir Arches*, whether we are treating solid or voussoir arches, uniform or isolated loading.

This remark applies also to many cases yet to be examined.

Now the loading being continuous in Fig. 8, the pressure line is a true curve

so that after finding the centers of pressure on the joints, a curve can be traced through them, by the aid of some paper curves described with about the radius of the arch.

We thus have a curve, which is almost exactly the equilibrium polygon p due to the continuous loading.

With this curve p, we proceed as before to find the true curve c. The arch ring was divided into 16 equal parts, and ordinates drawn through the center of each portion as in Fig. 6.

It is found that the true curve c, differs so little from the curve first drawn, that it may be taken for it, in investigating the stability against rotation of this arch. Thus the statement made in the former treatise, that "it seems highly probable, that the *actual* line of pressures is confined within such limiting curves, approximately equi-distant from the center line of the arch ring, that only one curve of pressures can be drawn therein, corresponding,

of the thrust in the limits taken," is verified in this instance.

The similarity in the curves so found, might have been anticipated from a consideration of the conditions $\Sigma M = o$, &c., especially the first.

The above principle will be found of utility in constructing trial curves of pressure. If, in any trial curve, the two conditions, $\Sigma M = o$, $\Sigma Mx = o$, happen to be realized, we see from the previous construction (Fig. 6), that the third condition $\Sigma My = o$ only involves a change in the horizontal thrust, together with a change in the length of ordinates, from the line k_1k_2 to the curve c which is readily effected.

37. The theory pertaining to solid arches is only applicable to voussoir arches, *when the actual curve of pressures lies in the inner third of the arch ring—the face of the voussoir being rectangular—with no mortar joints, the stones, moreover, fitting perfectly before the centers are struck.* It is approximately true for thin mortar joints

($\frac{1}{16}$ inch, say) that are allowed to harden before the centers are struck, the mortar being of the best quality.

The above seems to be the most exact solution of the stone arch for the cases assumed that has yet been proposed; though it seems scarcely necessary ot enter into it in testing the stability against *rotation* of such an arch, since this cannot happen theoretically, unless the arch ring is so small that only one curve of pressures can be drawn in it; so that if only one curve of pressures, even, can be inscribed, in the middle third say, although it may not be exactly the true one, yet it would indicate that the arch was stable, since it cannot fall until all of its cases of stability are exhausted. If, however, we desire to know whether the limit of elasticity has been passed in any voussoir, then the construction for solid arches, if applicable, had best be made; and the maximum strain sustained by any "fiber" be determined by the use of eq. (4), art. 11.

38. It is evident that *concrete* arches,

fall under the same treatment as solid arches of iron or other material. For such arches the question of *strength* is the only one necessary to consider, unless the *stability* of a pier or abutment is in question.

The spandrels of concrete arches, although generally built open, are nevertheless constructed as a part of the arch proper, the whole constituting one monolithic structure. The effect is to modify the previous construction somewhat, though possibly on the side of safety. The same remark applies to any spandrel bracing in iron, etc., arches.

LOADING—VARIATIONS IN EI, ETC.

39. Let us again refer to Fig. 6, and consider the character of the loading, as well as the design of the arch, with reference to the most accurate construction of the curve of pressures.

If the loads bear at the isolated points a_1, a_2, . . . , the moments at one of those points equals, $H \times \overline{ac}$, $\overline{ac}$ being the vertical distance between the two curves a and c at this point.

But the object is, to ascertain approximately at each point, a_1, a_2, . . , the average moment, M on the part of the arch s, of which that point is the middle.

The above construction is thus only approximately true, since the ordinates at the points a_1, a_2, . . are generally greater than correspond to the average M mentioned.

Now it is by no means necessary to suppose the loads as acting at the points a_1, a_2, . .; in fact, it is generally most convenient, to suppose the loads as acting at the same horizontal distance apart, as actually happens in most iron arches with open spandrels.

Even with continuous loading, the ordinates ac do not give a good average for the part s, when the loads are supposed to bear at a_1,a_2, . . , or at uniform distances apart measured along the arc, the true pressure curve passing slightly below the apices c_1,c_2, . . The above remarks of course apply equally and primarily to polygon p.

Therefore, it is generally best, to draw

polygon p (or curve p, if the loading is continuous) by supposing the loading applied at other points than $a_1, a_2, \ldots$; when the parts of the ordinates, drawn through $a_1, a_2, \ldots$, intercepted between vv' and curve p, will give the lines proportional to M_c more correctly than before. The residual small error can only be diminished by increasing the number of divisions of the arc, which remark applies in all cases.

40. If in place of dividing the arc into equal parts, we divide the *span* into equal parts, s in eqs. (7), (8), (9), &c., is no longer constant; so that if E and I are constant, the conditions for an arch fixed at the ends would become,

$$\Sigma Ms = o, \ \Sigma Mxs = o, \ \Sigma Mys = o.$$

Similarly for the supposed girder and arch acting as an equilibrium polygon.

If we divide the above equations by the horizontal distance between the loads, we see that each M must be increased in the ratio of the secant of the inclination at the point, in these, and the auxiliary equations, similar in form to the above. Thus

in Fig. 6, the ordinates of polygon p, as well as of curve a, must be increased in this ratio, after which the preceding methods apply in finding the position of mm' and k_1k_2, as well as the horizontal thrust.

This method introduces one advantage, with the several disadvantages: the consideration of moment areas in place of the single ordinates, since these ordinates are now proportional to the areas included between them and the equilibrium polygon.

41. If the quantities E and I of eqs. (7), (8) and (9) are variable also, we may divide the *span* into equal parts, and after drawing polygon p, alter the ordinates from the closing lines of p and a in the ratios, $s \div EI$, or in ratios proportional to these variable ones.

42. Now it frequently happens that the arch is increased in size towards the abutments, on account of the increased strain as we near the springings. If the ratio $s \div EI$ is constant for each length of the arch, having the same horizontal pro-

jection, then it is not necessary to alter the lengths of the ordinates of curves p and a, and the previous construction holds, when the span and not the arch is divided into equal lengths. This hypothesis ($s \div \mathrm{EI}$ constant) was adopted by Prof. Greene in his analytic treatment of the parabolic arch in *Engineering News* for 1877, and is the basis of all graphical constructions founded on a division of the span into equal parts when the ordinates p and a are *not* elongated, as note Prof. Eddy's treatment in "New Constructions in Graphical Statics." For very flat arches, even when E and I are constant, this method is sufficiently correct. It will be illustrated further on.

43. When E and I are variable, in place of the construction of art. 41, we may divide the *arc* into parts of such length, that $s \div \mathrm{EI}$ is the same for each length.

The division of the neutral line of the arch may be made by trial. Thus lay off a length s from the crown; change this (generally an increase) for the next

length in the ratio of EI for the two portions; similarly for other portions. If the last division does not exactly reach the springing, but leaves a small interval d, we should increase the length of the first division by $\frac{ds}{S}$, where s represents the length of the first division, and S the length of the arc from the crown up to the interval; each division is likewise augmented in this same ratio of $\left(s+\frac{ds}{S}\right)$ to $s=1+\frac{d}{S}$, the changes in EI for the same division being disregarded. The total increase is thus d, so that the last division just reaches the springing; and the arc is now divided so that, $s \div$EI is the same for each length. In arches, the sizes of the pieces do not vary so much, that the variation of E need be regarded, however different this may be in continuous bridges (see Bender's *Continuous Bridges*). It was included above, so as to present the subject in its most general form. We thus see how remarkably general this graphical method is:

enabling us, with the same ease, to treat any form of arch, with varying section or otherwise, loaded in any conceivable manner.

44. The positions of the live load, causing maximum strains in the flanges and web of an arch, can be found by treating separately each weight and then combining, for any piece, the maximum strains that can be sustained by it for any disposition of the load. For a single weight the pressure curve or polygon becomes two straight lines as in Fig. 5, so that polygon p in Fig. 6 is quickly drawn, when the construction proceeds as before.

It will be observed that for the same arch the positions of T and T′ are the same for each weight, so that it is unnecessary to draw the polygon 1′2′..8′ but once. Similarly in Fig. 7, the construction for the curve a, holds for all the weights, as also the position of the closing line k_1 k_2 (Fig. 6); so that the constructions are thereby simplified.

FLAT ARCHES.

45. When the arch has a small rise compared with the span, it is evident that on dividing the *span* into equal parts, drawing ordinates, &c., that these ordinates will almost coincide with those found by dividing the arc into equal parts as previously done. When this is the case, the following neat construction applies:

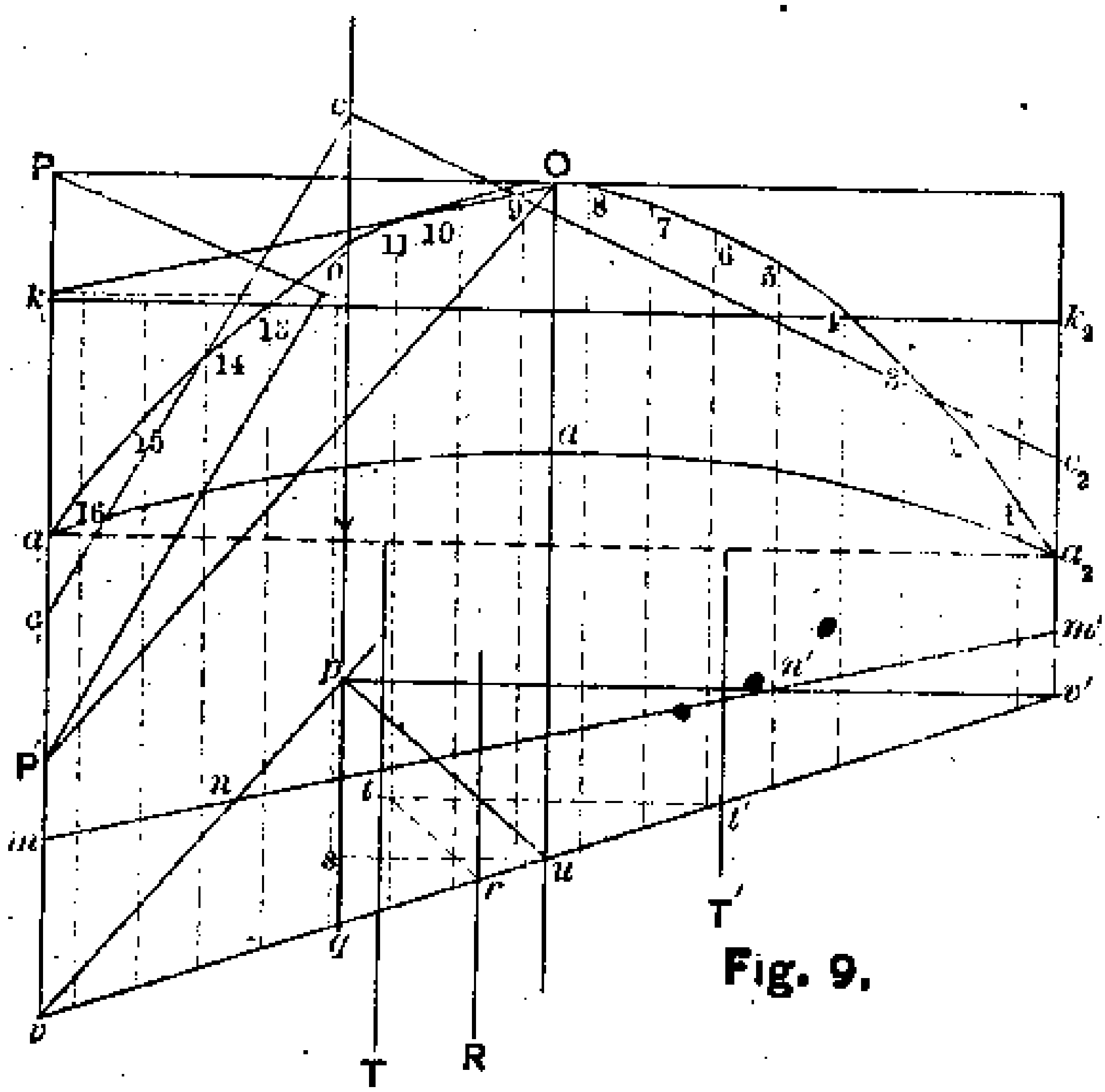


Fig. 9.

Let a_1 a a_2 (Fig. 9) be the neutral line of a circular arch rib of 68.'404 span, 6.'03 rise and 100.' radius, the half central angle being therefore 20°. Divide the span into 16 equal parts, and at the middle of every division erect ordinates as shown by the dotted lines.

The arch is so flat that it is best to increase the length of the ordinates, say four times, so that the neutral line takes the position 1, 2, ... This simply involves a corresponding decrease of $\frac{1}{4}$ in the pole distance in curves a and c according to art. 4, as we shall see more clearly as we proceed.

Let a single weight act at c, represented in intensity by the vertical PP'. Assume some point as o for a pole; then from some point v in the vertical through a_1 draw $vp \parallel o\mathrm{P}'$ to intersection p with the vertical through c, then $pv' \parallel o\mathrm{P}$ to intersection v' with the vertical through a_2; also draw the straight line vv'. We have first to find the true closing line mm' from the two conditions $\Sigma \mathrm{M}_c = o$, $\mathrm{M}_c\, x = o$. Now since the span has been

divided into equal parts, the ordinates from mm' to polygon vpv', that are proportional to M_c, are also proportional to the *areas* of the trapezoids of which they may be considered the medial lines, the horizontal altitudes, h being supposed the same. Now if we suppose the number of ordinates indefinitely increased, $h\Sigma M_c$ above and below mm' approach the actual areas of the triangular spaces included between mm' and vpv' as near as we please. Hence the most accurate result attainable by the graphical method, in this case, is when we employ moment areas in place of the corresponding ordinates. The same remark applies to curve a regarded as an equilibrium polygon.

46. Reasoning as before, the condition $\Sigma M_c = o$, indicates that the sum of the areas vmn and $v'm'n'$ must equal the area npn'. Adding to both sides of this equality, area $vnn'v'$, we deduce, area $vmm'v'$ = area vpv'. Conceive a line drawn from m to v' dividing $vmm'v'$ into two triangular areas. Their centers of gravity are on the verticals T and T',

drawn $\frac{1}{3}$ span from a_1 and a_2 respectively. Draw a straight line from p to the middle u, of vv'; the center of gravity of the triangle vpv' is at $\frac{1}{3}pu$ from u; the vertical R passing through it. The condition $\Sigma M_c = o$ is satisfied when the sum of the two triangles composing $vmm'v'$ is equal to the triangle vpv', and $\Sigma M_c x = o$ when the common center of gravity of the two triangles coincides with R, the reasoning being similar to that of art. 28.

These triangles having the same altitude, the span, are proportional to their bases vm, pq and $v'm'$. Therefore let pq represent the large triangle; from some point u as a pole draw uq and up; then, say from the intersection of qu with R at r draw rt and rt', parallel to up and uq respectively, to intersections t and t' with T and T′.

A line $us \| tt'$, divides pq into two parts, ps and sq, that are equal to the true values $\overline{mv}$ and $\overline{m'v'}$. This is evident, because if $\overline{ps}$ and $\overline{sq}$ are forces acting along T and T′, then by the construction of the equilibrium polygon trt', we see

that $R=pq$, acting through r is their resultant. Thus the first two conditions are satisfied. We may now draw a line through o parallel to mm', to intersection with PP′, which thus divides it into the two vertical components of the reactions.

47. We proceed now for the exaggerated curve of the arch 1, 2, ... exactly as before, and thus locate the position of the closing line k_1k_2.

In finding $\Sigma M_a y$ and $\Sigma M_c y$, as in art. 32, it may prove a convenience, to lay off the ordinates, proportional to M_a, or to M_c, along the line oP already drawn; then on drawing horizontals through 1 and 16, 2 and 15, etc., and choosing a pole as a, we proceed as in art. 32 to find $\Sigma M_a y$. This product having been found and laid off, the construction pertaining to it alone may be erased. On the line oP, or a parallel line, and the *same pole distance*, find the product $\Sigma M_c y$ as before. The moments M_c are proportional to the ordinates intercepted between mm' and vpv', as in the previous case.

It is of course well to ink all the lines

that are to be used again, for another position of the load, which can appropriately be done in red, or blue ink, to contrast better with the grey pencil lines.

48. From the construction last mentioned, we find that the ordinates of the equilibrium polygon p have to be elongated in the ratio of $k_1 c_1$ to vm; and the pole distance is decreased in the inverse ratio. Since the closing line $k_1 k_2$ is horizontal, we draw a horizontal through the point of PP′ that separates the vertical components of the reactions to o', the new pole distance.

On elongating the ordinates mv, $m'v'$, the proper amounts, and laying them off from k_1 and k_2 to c_1 and c_2 respectively, we find the pressure curve $c_1\ c\ c_2$, by drawing $c_1 c \parallel o'$P′ and $c\ c_2$, which last should be parallel to o'P. The point c may also be found by elongating the ordinate at p to mm' and laying it off above $k_1 k_2$ to c. On drawing lines from P and P′ parallel to $c_1 c$ and $c c_2$, we find the pole o', which should agree with the first determination; though very slight errors of this kind may be neglected.

The ordinates may be changed by proportional dividers if at hand, or in the usual geometrical manner. When the lines are inconveniently short or long, take multiples or fractions of them. If one of the ratio lines is changed in the same proportion as those laid off along it, the result will be the same, in finding the true distance on the other ratio line.

49. The previous construction gives the reactions $o'P'$ and $o'P$ acting at their points of application c_1 and c_2, which is all the data required to ascertain the strains in the arch due to the load at c, as we shall see further on. The actual reactions for the real arch a_1aa_2 correspond to a horizontal thrust four times as great as the above, the distances $\overline{a_1c_1}$, $\overline{a_2c_2}$ being diminished one-fourth, and are given below for various positions of the load, as found from a large scale drawing of 4feet to the inch.

Let a denote the half central angle, or the angle between the radii at a_1 and a, or a_2 and a. Let β denote the angle

from the vertical radius through a to the load at the neutral axis, measured at the center of the arc.

In the above figure we have taken the isolated weight PP′ equal to 10 tons, $a=20°$, and radius$=100$ feet, span 68.4 and rise 6. feet of the center line of rib.

We find by careful construction on the large scale drawing, the following values, for the vertical component V of the reaction farthest from the load, the horizontal thrust H and the distances a_1c_1 and a_2c_2, these distances being plus when laid off above a_1 or a_2.

β	V	H	$\overline{a_1 c_1}$	$\overline{a_2 c_2}$
	Tons.	Tons.	Feet.	Feet.
0°.	..5. ...	..27.08..	+0.87..	+0.87
	(5).	(26.69)	(0.82)	(0.82)
4°.	..3.53..	..24.60..	+0.1 ..	+1.39..
	(3.5)	(24.56)	(0.2)	(1.36)
8°.	..2.16..	.18.8 ..	−1.25..	+1.8
	(2.13)	(18.7)	(1.32)	(1.74)
12°.	..1.1 ..	..11.8 ..	−3.40..	+2.15
	(1.0)	(10.8)	(3.96)	(2.02)
16°.	..0.29 .	.. 3.4 ..	−11.0 ..	+1.82
	(0.27)	(3.36..	(12.0)	(2.25)

The numbers in parenthesis give the corresponding values obtained from Winkler's table mentioned below. We thus perceive in this case what reliance may be placed in a graphical solution, which was quickly made and not revised.

There were only 16 ordinates drawn, as in fig. 9; but with a greater number the result may be made as accurate as we wish, although the labor of construction is increased very much by using a large number of ordinates. Still there seems no other way of reducing error, especially for the loads near the abutments. The use of moment areas, as shown in Fig. 9, involves but little approximation in principle for flat arches but in establishing the closing line $k_1 k_2$ and the equality $\Sigma M_a y = \Sigma M_c$, we are forced to use the ordinates in place of moment areas, which process involves an approximation which is nearer the truth the greater the number of ordinates.

50. Having found the reactions at the abutments for a number of loads, the resultants of these reactions must be in

equilibrium with all the loads; so that if we find their position at the abutments, on combining one of these resultants with the loads, the final resultant should be equal and directly opposed to the resultant found at the other abutment. Moreover, on the principle that the moment of the resultant is equal to the sum of the moments of its components, *the moment at any point of the arc is the same whether we find the moments due to each single load, as above, and combine them, or whether we combine the reactions for the several loads into one, at each abutment, draw the resulting pressure curve and find the resulting moment at the point of the arch considered.*

To find the point of application of this resultant reaction at one abutment, resolve each reaction due to a single weight into vertical and horizontal components. The former pass through the end of the arch, and produce no moments about it. Hence taking moments about this point, we find the point of application of the

resultant reaction below or above the springing, equal to the sum of the moments of each horizontal thrust divided by the sum of the horizontal thrust. If now we draw the pressure curve from the point thus found, the horizontal thrust being equal to the sum of the thrusts due to each single weight, *this new pressure curve will satisfy the conditions of an arch fixed at the ends:* for the separate pressure lines for each weight satisfies the conditions $\Sigma m = o$, $\Sigma mx = o$, $\Sigma my = o$, $\Sigma m' = o$ &c., m, m', &c., being the moments at the points $a_1, a_2, \ldots$; hence the resultant pressure line will satisfy the conditions $\Sigma M = o$ &c., where M at any point is equal to the sum of the m's at that point (as just shown), as we find by simple addition of the eqs. for the separate weights. It is on this principle that we are enabled to consider separately the stresses due to loads, temperature etc., etc.

51. We see from the foregoing table that the horizontal thrust diminishes as the load is nearer the abutment.

It is well to assume such poles, that the ordinates M_c are of a convenient length; the pole being taken nearer the abutment as the load approaches it.

52. For the *exact* analytical treatment of the *circular* arch, the reader is referred to DuBois' *Graphical Statics*, pp. 271 to 311. The exact formulae for the distances $\overline{a_1c_1}$ and $\overline{a_2c_2}$ are given on p. 297, being due to Winkler.

On substituting the values of M_0, H and V taken from Winkler's tables, the term $k = \dfrac{I}{Ar^2}$ (where I is the Moment of Inertia of the constant cross section, A its area, and r the radius of the circle) being disregarded, we have the following table for the quantities $\overline{a_1c_1}$ and $\overline{a_2c_2}$, distances measured above the springs being plus. In the table, α is the half central angle, and β the angle from the crown to the load given in terms of α; h is the rise, all referring to the neutral line of the rib:

Values of $c_1 = \overline{a_1 c_1}$

β	$a=10°$	$a=20°$	$a=30°$	$a=40°$
0	+ .1343	+ .1370	+ .1417	+ .1488
.2	+ .0008	+ .0034	+ .0072	+ .0142
.4	− .2217	− .2196	− .2167	− .2106
.6	− .6667	− .6655	− .6646	− .6621
.8	−2.0020	−2.0015	−2.0148	−2.0142
a	h			

Values of $c_2 = \overline{a_2 c_2}$

β	$a=10°$	$a=20°$	$a=30°$	$a=40°$
	+	+	+	+
0	.1343	.1370	.1417	.1488
.2	.2232	.2262	.2307	.2381
.4	.2869	.2896	.2943	.3019
.6	.3345	.3371	.3422	.3493
.8	.3716	.3748	.3795	.3826
a	h			

Each of the members in the table must be multiplied by h, the rise of the center line.

The "approximate formulæ" given (p. 261) for $\overline{a_1 c_1}$ and $\overline{a_2 c_2}$, do not agree with these nearly exact values (k being very small) and are prob-

ably erroneous,* as might have been anticipated; since the term involving k is of undue importance. With very extensive tables (which would of course involve great labor in preparation) the solution of the solid or braced arch becomes a very simple matter. By interpolation, the preceding tables may be made to do good service. In the chapters just quoted is also the equation of the locus of the point c (Fig. 9). With the points c_1, c and c_2 thus found, the reactions are determined completely, so that the complete solution of the solid or braced arch is thus effected with great ease and rapidity by this combination of graphical and analytical methods.

53. Having found the reactions and their point of application for each single weight acting on the arch, we find, as in art. 11, or art. 82, further on, the strains in every division s, both of flanges and web and tabulate them. From such a table, we readily find the position of the

* I have verified the *exact* formulae given in the first four chapters of Supplement to Chap. XIV of DuBois' *Graphical Statics* [save that B_2 on page 284 should equal $(4\,a \cos^3 \alpha)$ in place of $(2\,a \cos^3 \alpha)$], but not the tables; so that I cannot vouch for the last two tables given, though I believe the tables on which they are founded to be correct.

live load that causes maximum strains in any part of the arch, and can thus tabulate them. For illustrations of this method of treatment see DuBois' *Graphical Statics*, p. 374 and on.

For the voussoir arch, this determination of maximum stresses is not of so much importance as finding those positions of the loads that cause a maximum departure of the curve of pressures from the center line at various points. Prof. Greene, in *Engineering News* for 1877, p. 178, has given a table of actual moments at 19 points of a solid parabolic arch, due to a single load, placed successively at each point of division. As he states, "the greatest possible positive M, as well as the greatest possible negative M, for any combination of weights, occurs at each abutment; positive maximum when the span is loaded from the other abutment to and beyond the center one or two points" (*i.e.*, $\frac{1}{20}$ to $\frac{2}{20}$ of the span); "negative when the other portion of the span is covered." This position of the load for the circular arch may be inferred from the preceding tables for c_1 and c_2.

In a parabolic voussoir arch, in which no joints open &c., we must therefore assume the uniform live load as extending from one abutment about six-tenths, or slightly less, of the span, as an approximation; though it is possible that other positions of the live load corresponding to a less hoizontal thrust, may cause the pressure curve to depart even farther from the center line, than the position just mentioned. This can only be determined by trial.

The same remarks apply to flat circular arches: in fact it may be well to test any style of arch by first supposing the live load to cover about half the span from the abutment.

TEMPERATURE STRAINS.

54. It is usual to class under this head, strains due to a change of temperature above or below the temperature at which the arch is finished, (supposing all parts then to fit accurately); as well as strains due to the components T(art.9) tangential to the rib that compresses it,

and cause it to be suited to a less span than it is fitted to; together with strains due to an actual lengthening of the span due to the compression.

Denote the temperature at which there is no strain in the arch due to temperature, as the *mean* temperature; and the greatest deviations from this, above or below, by $+t$ and $-t$.

Denote the expansion of the metal for a unit of length and one degree by ε. The total change of span is then $l\varepsilon t$, denoting the span by l.

This tendency to a change of span being resisted by the abutments, causes a horizontal thrust or tension there. Since for the *arch fixed at the ends*, in direction, &c., there will also be strains in the flanges at the abutments due to this change, there must also be a couple at the abutments to produce this effect.

Thus in Fig. 10, representing half of the arch, suppose a horizontal force H acts at k; conceive two horizontal and opposed forces, each equal to H to act at a. This does not disturb equilibrium, but

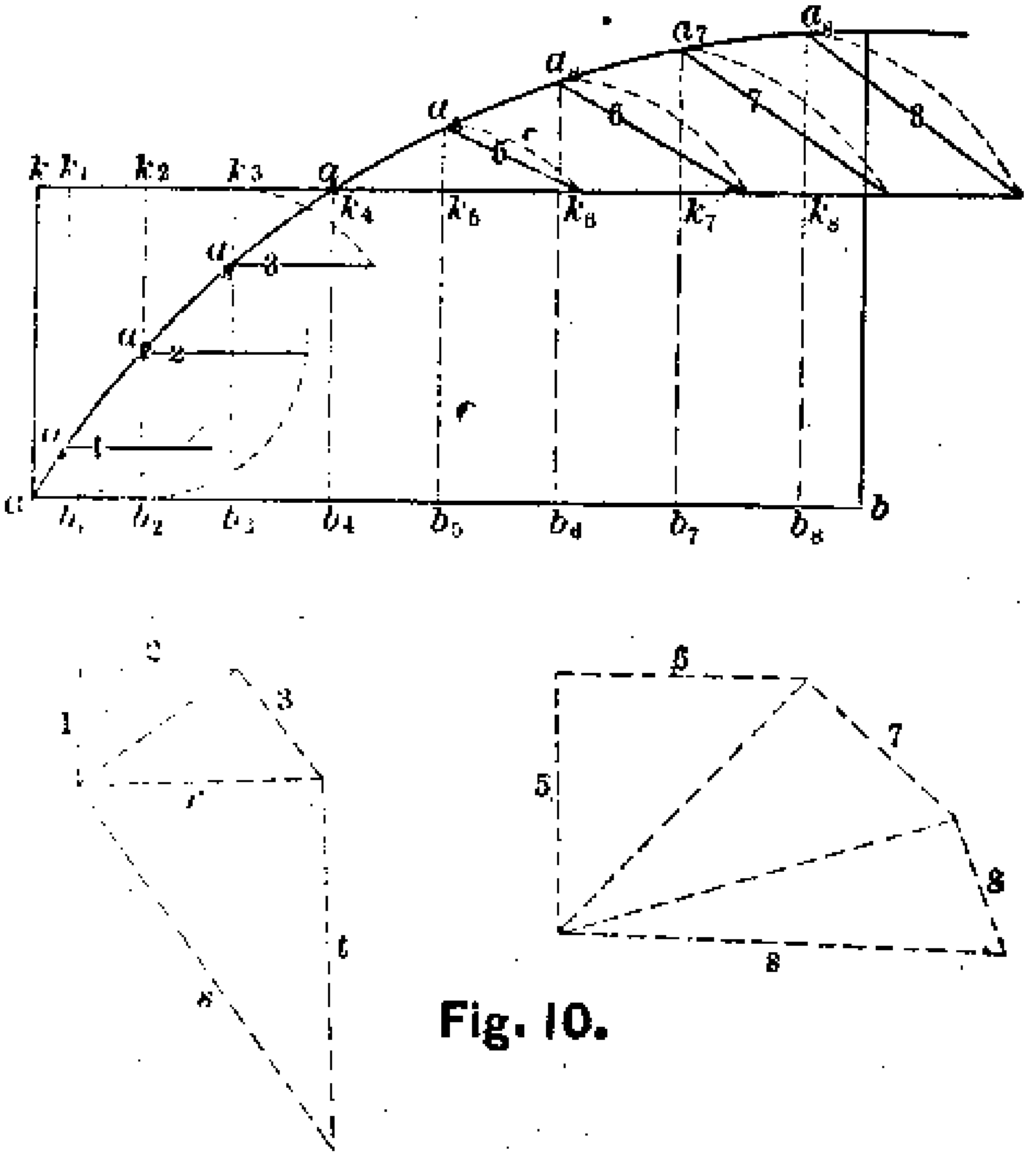


Fig. 10.

effects a transfer of H to a (where it must be sustained) and adds the couple H.$\overline{ak}$. If the arch is symmetrical about the center, the same condition of affairs must exist at the other abutment, so that the force at k must act along the horizontal kk_8, as was assumed.

This follows again from the condition used to fix the line kk_8, which is, that the

total change of direction of tangents, in going from one abutment to the other, is zero; *i.e.*, if E, I and s are constant, we must have

$$\Sigma M = o,$$

which gives the same position to the line kk_8 as found in art. 31.

Therefore having divided the *half arc* into any number of equal parts (8 in the Fig.) and drawing ordinates a_1b_1, a_2b_2, . . through the centers of each part, we have the moment at any point, as $a_1 = H.\overline{a_1k_1}$, etc; the moments changing sign on opposite sides of kk_8.

55. The object now is to find H.

The effect of the change of temperature is to produce a virtual alteration of span (this being small), so that (eq. 8) of art. 13 must be satisfied. We have then,

$$h = l\varepsilon t = 2\frac{s}{EI}\,\Sigma My;$$

the summation ΣMy being taken from the center to one end, it being the same for the other half of the arch.

The moments M, at the successive

points a_1, a_2, . . , we have just seen, are equal to products of the type (H.$\overline{ak}$), so that the preceding eq. becomes

$$\mathrm{H}\Sigma(ak.y)=\frac{h\mathrm{EI}}{2s}$$

If the known quantities h, I, s, $\overline{ak}$ and y are given in feet (say) in true or real dimensions, and E in tons (say), then on forming the above sum, and solving the eq., we find the value of H in tons. If E is given in pounds, then H is likewise.

56. The sum of the products $\Sigma\overline{ak}.y$, may be found by calculation or graphically as previously shown. The following familiar graphical construction (see Chauvenet's *Geometry* pp. 120, 137) is a good one when the segments are not too dissimilar in length.

1/. For the part below kk_5, describe circles on $\overline{k_1b_1}$, k_2b_2, . . . as diameters; the intersections of the horizontals through a_1, a_2, . . with these, give the lines 1, 2, . . , which measured to the scale of the arch and squared, give the products $\overline{a_1k_1}.\overline{b_1a_1}$, $\overline{a_2k_2}.\overline{b_2a_2}$, etc. Therefore in the Fig. just

below this part of the arch, draw 2 perpendicular to 1, of the lengths above; the square of the hypotenuse of the right triangle is equal to 1^2+2^2. Next draw 3 perpendicular to this hypotenuse, we have then $r^2=1^2+2^2+3^2$.

2/. Next consider the part of the arch above kk_8. On $\overline{b_5a_5}$, b_6a_6, . . ., as diameters describe circles; from their intersections with kk_8, draw lines to a_5, a_6, . . .; and denote the lengths of these lines by 5, 6...; then as before find s^2=the sum of their squares.

3/. Draw a perpendicular t to r of such length that s is the hypotenuse of the right triangle whose three sides are r, s and t. Then $t^2=s^2-r^2=\Sigma \overline{ak}.y$. Hence measure t to the scale of the arch and square the number found, which square is thus equal to $\Sigma \overline{ak}.y$ required, whence H may be found from the preceding equation.

Example.—Let the span of the arch be 518 ft., the rise 51.8 ft., EI=39,680,000 foot tons, and $l\varepsilon t$=0.2735 feet (ε is taken as .000012 for 1°c), t being 44°c (about 80° Fahrenheit).

On dividing the semi-arch, drawn to a scale of 20 feet to the inch, into sixteen equal parts, and proceeding as before, we find $t^2 = \overline{62.8}^2 = 3944. = \Sigma \overline{ak}\cdot y$, whence $H = 82.4$ tons.

The analytical formula (see Du Bois' *Graphical Statics*, p. 302, eq. (66), omitting the term k, which is extremely small (less than $\frac{1}{15000}$ for the middle span of the St. Louis Bridge) gives $H = 81.9$ tons, on substituting in it, $a = .3947353$, $\sin a = .3845639$, $\cos a = .9230984$ and $r = 673.4$; a in degrees is put at $22°37'$.

57. When E, I and s are variable, we may pursue the treatment of art. 41; *i.e.*, alter the distances $k_1 a_1$, etc., for a trial line kk_8 in the ratios $s \div EI$. After the position of kk_8 is obtained we proceed as before, using the altered ordinates.

In fact, the preceding equation must now be written,

$$h = 2H\Sigma\frac{\overline{ka}.y.s}{EI};$$

$\overline{ka}$ being the ordinates to the arch from the closing line. Now we see from this equation, that at each point a_1, a_2, . . , that

$\overline{ka}$ being multiplied by $\frac{s}{\mathrm{EI}}$, changes the ordinates in the ratio before stated.

Now if EI is greatest, nearest the abutments, the changed ordinates *ka* become less there proportionally; so that the line kk_8 is raised above its normal position, whence the Σ above may be less than before and H greater. This was the case at the St. Louis bridge, the true horizontal thrust due to temperature being 104 tons, the varying cross section being considered.*

58. We have seen that the strains due to a change of span, real or virtual, result from a horizontal force H at *a*, and a couple H.$\overline{ak}$ also acting at *a*. A rise of temperature, producing a virtual shortening of the span, the force H at *a* acts towards *b*. On the contrary, for a fall of temperature below the mean, or the elastic shortening of the arch due to the tangential forces, or for an actual lengthen-

* The entire construction above for finding H differs essentially from that given by Prof. Eddy in *Researches in Graphical Statics*, which is believed to be erroneous, as also the determination of temperature strains for the other arches treated by him.

ing of the span the force H at a acts towards the left. The couple $H.\overline{ak}$ is in the first case right handed; in the second case left handed. We see easily that the moment, at any point a_2

$$=H.\overline{ak'}-H.\overline{a_2b_2}=H.\overline{k_2a_2},$$

as was asserted. In fact we must conceive two opposed horizontal forces each equal to H, at a_2, one of which, with the force H at a forms the couple, whose moment is $H.\overline{a_2b_2}$ as above. This gives us finally the moment $H.\overline{a_2k_2}$ at a_2, together with a single horizontal force H, acting to the right or left according as H at a acts to the right or left. This force can be resolved into tangential and normal components at a_1, a_2, etc.; from whence the final stresses can be found as usual. If more convenient a single force H, acting along kk_2 may be used, being the resultant in position of all outward forces acting on the arch, and the strains in the flanges determined by taking moments about suitable apices.

59. If preferred, the force H due to

temperature, compression, &c., may be combined with the reactions due to any system of loads, according to the principles of art. 50; the resulting pressure curve, with a pole distance equal to that due to the loads, plus H, is then the true one by which to determine the strains in any member of the arch. In this way is included the effect of the components T (art. 9), hitherto neglected. The sum of the horizontal projections of the shortening of each part s of the arc, due to T, which is easily computed is the virtual change of span.

If the strain f, per square unit, due to T alone, is supposed approximately the same on each cross-section, the change of span is $\left(\frac{f}{E} L\right)$, where E is the modulus of elasticity and L the span, from which the resulting strains are readily found. This is generally a sufficient approximation.

The strains due to temperature, change of span, &c., are very large, requiring the most accurate fitting of

the arch members. By diminishing the proportionate depth of arch; or using a material, as steel, with a high modulus of elasticity, these strains may be diminished.

60. We have treated the arch "fixed at the ends" with some detail, since its solution includes, to a great extent, that of arches with hinges. The latter will now be considered, as briefly as is consistent with clearness, avoiding, as much as possible, repetitions of the same con struction.

ARCH FIXED AT ONE END AND HINGED AT THE OTHER.

61. Let Fig. 11 represent an arch fixed in direction at B and jointed, or free to turn, about the point O at the other abutment. We have taken the point O outside of the neutral line *aa*, as this presents the most general case. In fact, when for an arch *with flat joints* at both ends, for any loading, the pressure curve passes outside of the arch at one abutment, when treated as an arch fixed at

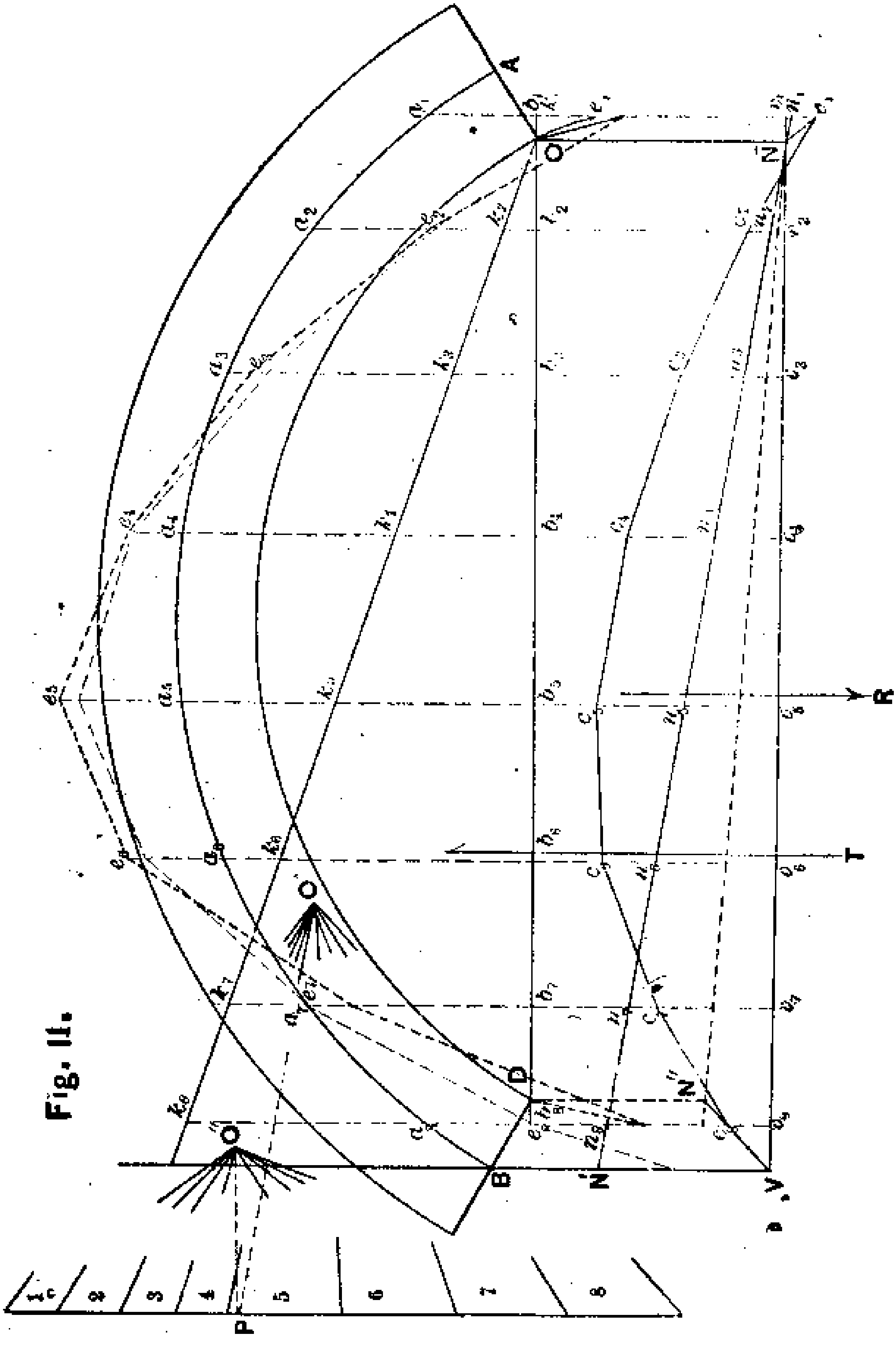


Fig. II.

the ends, the arch there will rotate about the edge of the joint, at O, *nearest the pressure curve*; and must be treated as an arch fixed at B and jointed at O. If the pressure curve thus found, however, passes outside the joint at B, then the arch must be regarded as hinged at both ends for the load in question. If the pressure curve for the arch fixed at the ends passes outside of both joints B and A, we next try the pressure curve for the arch fixed at one end (the end where the pressure curve is nearest the neutral line), and if the resulting curve again leaves the arch ring at the end supposed fixed, then the arch falls under the head of the "arch hinged at both ends," the points free to turn, lying on the abutment joints nearest the pressure curve. (See Mr. Pfeifer's article in VAN NOSTRAND'S MAGAZINE for June, 1876, p. 494.) If the arch is firmly bolted to the abutments at the ends, so that the connection can exert sufficient tensile resistances, as in the St. Louis bridge, there can be no *rocking*

properly treated as fixed at the ends in all cases. It is evident how convenient the graphical method is in determining the special case to be treated even when the analytical method is finally applied.

62. The arch, Fig. 11, is taken of the same proportions and divided up in the same manner as Fig. 6. The true pressure curve e is shown by the dotted line, the force polygon being O″18 on the left. We shall presently show how to draw this curve. Granting that it is the true one, it possesses some peculiarities, well to note, about the point o, through which the right reaction of all the loads must pass in the direction e_1O. On combining this reaction (ray 1 P on the left), with the load at a_1, we get the resultant (ray 12), acting along e_1 e_2, on the part of the arch a_1a_2 and so on. Hence e_1e_2 does not pass through O.

We draw the equilibrium polygon c, as in Fig. 6, with an assumed pole o', except that at c_1, the direction of the reaction, or ray 1 P, is produced to intersection N with the vertical through O.

If we draw NN′ as a trial closing line, and suppose forces $F=o'P$ ($o'P$ being parallel to NN′) applied at N, N′, c_1 and V, as in Fig. 5 parallel to NN′, we see that ccNN′ is the equilibrium polygon due to the weights, with moments at B and a_1 proportional to VN′ and $n_1 c_1$. It is seen that at each apex, N, c_1, c_2, .. V, N′, we have forces in equilibrium.

On conceiving the polygon cut about N, &c., as in art. 4, we see that the moment at N is zero, as we desire it to be.

The moment at B, necessary to fix the end of the arch, acting as a girder, is equal to H.$\overline{N'V}$, H being the horizontal thrust corresponding to a pole o', and $\overline{N'V}$ having its proper length, to be determined presently.

Now the true pressure curve of the arch must satisfy the conditions of art. 15,

$$\Sigma_B^A Mx=o, \quad \Sigma_B^A My=o;$$

the origin of co-ordinates being at the point O, x horizontal, y vertical.

63. We must likewise have for the "girder," since the vertical deflection of

o below the tangent at B is zero, $\Sigma M_e x = o$; whence $\Sigma M_a x = o$, as in art. 23. Now M_e is proportional to ordinates of the type $\overline{nc} = (vc - vn)$; so that the preceding condition may be written, $\Sigma\, (\overline{vc} - \overline{vn})\, x = o \;\; \therefore\; \Sigma\, (\overline{vc} . x) = \Sigma\, (\overline{vn} . x)$. That is, calling the horizontal distances of $c_1, c_2, ..$, from $o, x_1, x_2, ..$, we must have,
$(\overline{v_1c_1}.x_1 + \overline{v_2c_2}.x_2 + ..) = (\overline{v_1n_1}.x_1 + \overline{v_2n_2}.x_2 + ..)$
It is seen that although $\overline{v_1c_1}$ and $\overline{v_1n_1}$ are minus, yet x_1 is also minus (art. 12) so that all quantities in the above equation may be regarded as plus; so that the result is the same if $\overline{v_1c_1}$ and $\overline{v_1n_1}$ are supposed laid off a distance x_1 to the left of N, in the above summation. We can form the above products graphically as shown in art. 32, the parallels being already laid off, and if the two members of the equation are not equal, we have simply to alter VN′ in the ratio of the right to the left members, which thus fixes the position of the line NN′.

Otherwise, by aid of two equilibrium polygons, using $\overline{vc}$ and $\overline{vn}$ as forces,

(v_1c_1 and v_1n_1 being laid off to left of N as above) find their resultants T and R similar to T and R of Fig. 6; then if t and r are their horizontal distances to N, we must have, $Tt=Rr$. If these products are unequal, alter VN′ as before. This method is best in view of what follows.

64. For the curve aa regarded as an equilibrium polygon, some line kk, passing through O must be the closing line; the ordinates of the type $\overline{ka}=\overline{ba}-\overline{kb}$ are then proportional to M_a, so that as before we must have the following condition satisfied, the quantities all being plus,

$$(-\overline{b_1a_1}.x_1+\overline{b_2a_2}.x_2+\ldots)$$
$$=(\overline{b_1k_1}.x_1+\overline{b_2k_2}.x_2+\ldots)$$

These products, except the first, are all plus, and may be formed as above. Since the effect in the summation is the same if we consider b_1k_1 laid off, x_1 to left of O, all quantities being plus, we can use the second method above more easily in this case; for T′=sum of ordinates of type bk, acts in the same vertical as T, so that if the first construction

has been made, we have only to find T′, when $T't$ will be known. The resultant, $R'=\Sigma(ba)$, acts through the crown, since b_1a_1, b_2a_2, . . , are all positive; the lever arm of R′ about o, r', equals the distance from the center of the span to o.

We should have, if k_1k_8 is in its proper position, $T't=R'r'$; otherwise alter $b_8 k_8$ in the ratio of $R'r'$ to $T't$, to find the true position of the closing line k_1k_8.

65. From the condition that the space is invariable we have,

$$\Sigma My=\Sigma(M_c-M_a)y=o$$
$$\therefore \Sigma M_c y=\Sigma M_a y;$$

the summation being extended over the entire span.

Writing the proportional equation in full, we have, in this case, putting $y_1=b_1a_1$, $y_2=b_2a_2$, etc.,

$$(-\overline{n_1c_1}\cdot y_1+\overline{n_2c_2}\cdot y_2+\overline{n_3c_3}\cdot y_3+\overline{n_4c_4}\cdot y_4$$
$$+\overline{n_5c_5}\cdot y_5+\overline{n_6c_6}\cdot y_6-\overline{n_7c_7}\cdot y_7-\overline{n_8c_8}\cdot y_8)$$
$$=(\overline{k_1a_1}\cdot y_1+\overline{k_2a_2}\cdot y_2+\overline{k_3a_3}\cdot y_3+\overline{k_4a_4}\cdot y_4$$
$$+\overline{k_5a_5}\cdot y_5+\overline{k_6a_6}\cdot y_6-\overline{k_7a_7}\cdot y_7-\overline{k_8a_8}\cdot y_8).$$

By way of variety, let us scale off the

above distances from the drawing, using a scale of 40 feet to the inch.

On forming the sum of the products indicated we have for the ratio,

$$\Sigma M_a y \div \Sigma M_c y = \tfrac{42}{15}$$

Therefore each of the ordinates nc of the polygon c must be increased in this ratio; after which they may be laid off from the closing line $k_1 k_8$, and the true pressure curve drawn as shown by the dotted line ee.

66. In order to find the new position o'' of the pole, for the true force polygon $o''18$ on the left, we draw from the assumed pole o', a line $o'P \parallel NN'$, thus dividing the load line, 18 into the two vertical components of the reactions. Then draw the horizontal Po'' equal to the old pole distance diminished in the ratio of 15 to 42, giving o'', the new pole, from whence the pressure curve may be tested, or drawn anew having one starting point.

In altering the length of lines in a given ratio, use may be made of proportional dividers, for small drawings, espe-

cially; or the common geometrical method of using two ratio lines may be employed, as explained before.

As a final test of this pressure curve, we find by scaling the ordinates, multiplying, &c., that $\Sigma Mx = o$, and ΣMy is very nearly zero also. The ordinates to dotted curve above aa are of opposite sign to those below in this summation.

67. On comparing the above pressure curve with that of Fig. 6, we see that it departs much farther from the neutral line aa, though the horizontal thrust is much less. The depth of arch ring, compared with its span, has been greatly exaggerated for usual cases in practice; so that the point O is generally much nearer A. When O is to the right of the vertical a_1b_1, the construction is, moreover, slightly simplified, as there are no negative ordinates, v_1c_1, v_1n_1, b_1k_1, or abscissas x_1, for this case. This is likewise true when O is taken above a.

For different loading or position of O, the form of the pressure curve may be materially altered.

For the determination of maximum stresses, we have to treat single weights as before, and combine their effects.

68. *Temperature Strains.*—A real or virtual alteration of span produces strains, similar to those caused by a force acting at O along k_1k_8; since such a force, whose horizontal component, call H, causes moments of the type $H.\overline{ak}$ at each point a; therefore the condition that the vertical deflection of O be zero, $\Sigma My = H\Sigma(\overline{ak}.y) = o$, requires the same closing k_1k_8 previously found, the moment at O being zero. Calling h the change of span due to temperature, compression, &c., we have as in art. 55,

$$H\Sigma(\overline{ak}.y) = \frac{hEI}{s};$$

from whence H may be found as before, noting that $\overline{ak}$ is minus on one side of k_1k_8, and plus on the other side in the summation.

69. For very flat arches, we may divide the span into equal parts, whence the ordinates corresponding to M_c and M_a are proportional to the areas of the trape-

zoids of which they are the medial lines, so that moment areas may be employed; which is most accurately done in connection with Simpson's rule. The reader is referred to Eddy's "Researches in Graphical Statics," for the most extended application to arches of moment areas.

The reader will do well to apply the principles of art. 34, in applying the two conditions, $\Sigma Mx = o$, $\Sigma My = o$: especially as a new form of equilibrium polygon is introduced. The results are of course the same in both cases. The method of that article may be applied to all the following cases.

ARCH HINGED AT BOTH ENDS.

70. Let the same arch, fig. 11, be regarded as hinged at O and D, the lower edges of the abutment joints. The reactions must now pass through O and D. Having drawn a trial curve cc, extend V_{c_n} to intersection N'' with the vertical through D; the line NN'' is the true closing line for curve c. Hence drawing through O' a line parallel to it, we divide

the load line 1.8 into the two vertical components of the reactions.

To elongate the ordinates of polygon c to their true lengths, we have the condition that the span is invariable,

$$\Sigma M_y = \Sigma (M_c - M_a) y = o$$
$$\therefore \Sigma M_c y = \Sigma M_a y = \Sigma (y^2);$$

the line OD being the true closing line.

On measuring $y = ba$, at each point $a_1, a_2, \ldots$, and the ordinates proportional to M_c included between NN″ and polygon c, noting that the ordinates at c_1 and c_8 are negative, and forming the above products we find that,

$$\frac{\Sigma M_c y}{\Sigma M_a y} = \frac{31}{103};$$

hence we must shorten the pole distance in this ratio, and lengthen the ordinates in the inverse ratio, as shown previously. The lengthened ordinates from NN″ to curve c, are laid off from $b_1 b_8$ giving the true pressure curve, as shown by the dotted line passing to the right of the first curve at e_6 and e_7 and crossing it about e_c and e_2. When the hinged, or

free to turn, points are at A and B there are no longer negative moments at c_1 and c_8.

71. *Temperature strains.* The thrust or pull H due to temperature &c., can only act along OD, giving moments of the type $H.\overline{ba}$; whence $Hs\Sigma(\overline{ba}.y) - hEI$ from which equation H may be found on substituting the values of $s, \Sigma(\overline{ba}.y) = 103$, h and EI as hitherto shown.

ARCH FIXED AT THE ENDS AND HINGED AT THE CROWN.

72. Let the curve AaB, Fig. 12, represent the neutral line of an arch-rib, fixed

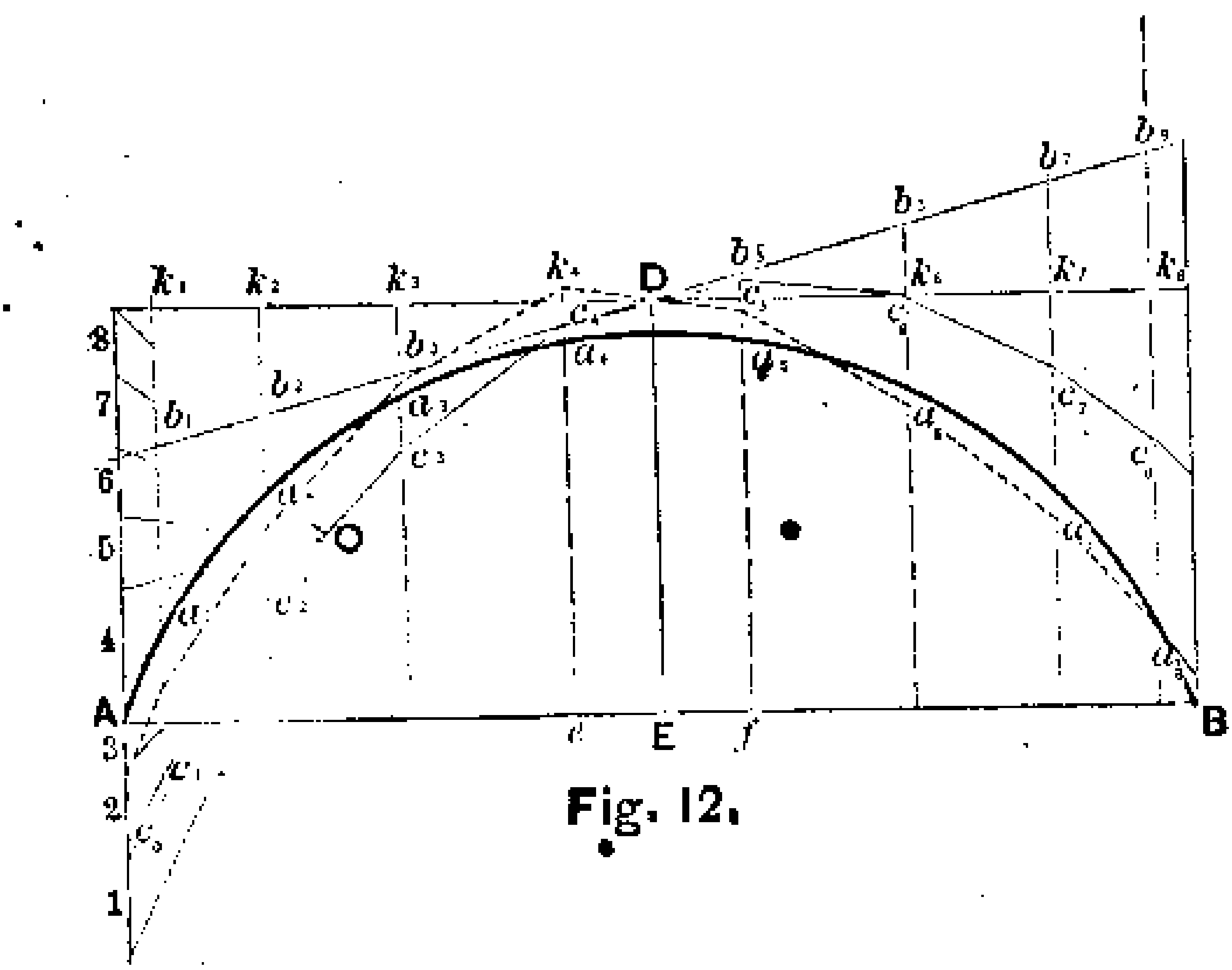


Fig. 12.

in direction and position at A and B, and hinged, or free to turn, at a point D above the crown, in the upper flange say, so that the pressure line must pass through D.

Divide the arc into 8 equal parts say, and draw ordinates through the middle of each part at $a_1, a_2, \ldots$, Lay off the loads 1, 2, . . . , acting at a_1, a_2, on the left vertical; assume a pole o and draw the equilibrium polygon *cc*. We have caused it to pass through D for convenience. If *bb* is a trial closing line, for the rib acting as a girder, the moments

at $a_1, a_2, .$, are proportional to $\overline{b_1c_1}, \overline{b_2c_2}, \ldots$,

Now by arts. 17, 19 and 25 we must have,

$$\sum_A^D (bc.x) = \sum_B^D (\overline{bc}.x);$$

(D being the origin of co-ordinates, x horizontal, y vertical) since the vertical deflection of D for the two halves of the arch is the same, owing to the hinge.

The condition above shows that the closing line *bb* must have such a posi-

tion, that if the ordinates b_1c_1, b_2c_2, . . . , b_8c_8, are taken as forces acting in position, their resultant must act through D; for the left member represents the sum of the moments of these supposed forces for the left half of the arch, which must equal a similar sum for the right half of the arch.

By aid of an equilibrium polygon or otherwise, find the position of the resultant of the supposed forces b_1c_1, b_2c_2, . ., and suppose it to pass through e, to the left of the verticle DE. Now lower slightly the left end of line $b_1 b_8$; find the resultant of the new forces $b_1 a_1$, etc., and suppose it to pass through f, to the right of E. If the left end of $b_1 b_8$ be now raised in about the ratio of $\overline{Ef}$ to $\overline{Ee}$ of the last change, its position will be very nearly correct. It may be tested, &c. as before. Suppose now, that b_1b_8 is the correct position of this closing line. It is evident that it is the best to fix the line b_1b_8 as nearly correct in the first instance as possible. A more precise method can be given, but the above is probably as direct as any.

73. The closing line kk of the arc AaB, regarded as an equilibrium polygon, must, of course, pass through D; and since the sum of the moments of k_1a_1, k_2a_2, . . ., to the left of D, must equal those to the right of D, as for the girder, the closing line must be horizontal, since the arch is symmetrical about the crown. The ordinates of the type ka are now proportional to M_a.

74. From the third condition, art. 17, which shows that the horizontal displacement of D for both halves of the arch is the same, though in opposite directions, we have

$$\sum_A^D (M_c - M_a)y = -\sum_B^D (M_c - M_a)y.$$

It is plain that if the left member is plus, that the sum in the right member must be minus, for the bending of one half of the arch is different in kind from that of the other half; hence we have added a minus sign to make the final result plus. From this equation we derive

$$\sum_{A}^{D} M_c y + \sum_{B}^{D} M_c y = \sum_{A}^{D} M_a y + \sum_{B}^{D} M_a y,$$

all ordinates being plus.

Now M_c and M_a are proportional to the ordinates of the type $\overline{bc}$, $\overline{ka}$, respectively; also at a_1, a_2, . . , y has the values $k_1 a_1$, $k_2\, a_2$, . . . , respectively, so that the preceding products are easily found graphically as in art. 7.

If the two members of the equation are not equal, we must change the lengths of the ordinates $b_1 c_1$, $b_2 c_2$, . . . to make them so, since the right member is constant. This involves an opposite alteration in the pole distance. Now lay off the altered ordinates from k_1, k_2, &c. downwards, to locate the true pressure curve, as shown by the dotted line.

On drawing through o a line parallel to $b_1 b_8$, we divide the load line 1 . . 8 into the vertical components of the reactions. Draw through this point of division a horizontal equal to the new pole distance to locate the new pole. These last lines are not drawn to avoid confusing the figure. The pressure curve may be tested

to see if ΣMy for both halves of the arch is the same. We have now the reactions in position, together with the pressure curve from which the strains can all be found.

75. *Temperature Strains.*—The moments caused by a real or virtual alteration of span, must cause the same to rise or fall at D for one-half of the arch as the other; so that k_1k_8, as previously determined from this condition, is the line along which the horizontal thrust H due to temperature, &c. acts. The moment at any point as a_2 is therefore $H.\overline{a_2k_2}$. We determine H exactly as in art. 55, noting that in the second equation of that article, that $\overline{ak}$ and y are equal for this case, the ordinates being measured from k_1k_8 downwards.

On measuring the ordinates of the type ka in Fig. 12 to scale, squaring and adding, we find the $\Sigma(\overline{ak}.y)$ of art. 55, whence the value of H may be found.

ARCH RIB FIXED AT ONE END, AND JOINTED AT THE OTHER END AND AT THE MIDDLE.

76. Let the arch, Fig. 12, be fixed at B and jointed at A and D. This is only a particular case of the preceding construction. Since there can be no bending moment at A or at D, the closing line k_1k_8 must pass through these points, so that its position is at once given. Suppose it drawn, and its intersections with the ordinates marked k_1, k_2, . . . , as in the figure; then we have only to determine the new pole distance for polygon c, with the resulting change in the length of ordinates bc. The closing line $b_1\, b_8$ must also pass through D, and C_0 must coincide with it.

77. Now the horizontal displacement of D for both halves must be the same though in opposite directions from their abutments; hence the preceding conditions, that may be written,

$$\sum_{A}^{D}(\overline{bc}.y)+\sum_{B}^{D}(\overline{bc}.y)=\sum_{A}^{D}(\overline{ak}.y)+\sum_{B}^{D}(\overline{ak}.y)$$

apply, the ordinates $\overline{bc}$ and $\overline{ak}$ being all

regarded as plus. (Prof. Eddy, in his treatment of this case, takes the differences of quantities, proportional to those in each member, in place of the sum as above. This is plainly an oversight, as we see from the connection with the preceding case). Proceeding now as before we locate the true pressure curve.

78. The thrust due to temperature must act along the closing line $k_1 k_8$, drawn through A and D. Call F the inclined force so acting, its horizontal component being H. The moment at any point a_2, is then H. $\overline{a_2 k_2}$, or of the type H.$\overline{ak}$; so that the principles of art. 55 apply, except that the moments either side of the center not being the same, we must find the sum of the products of the type $(\overline{ak}.y)$ for the entire arch, $\overline{ak}$ being of different signs for the two halves of the arch.

$$\therefore \; \mathrm{H}\sum_{\mathrm{B}}^{\mathrm{A}}(\overline{ak}\cdot y) = \frac{h\mathrm{EI}}{s}$$

The ordinates y are still reckoned from the horizontal through D, one of the hinged points, vertically downwards.

We find H from the above equation, as illustrated in art. 56, from whence the strains due to the real or virtual change of span h are readily found.

ARCH WITH THREE HINGES.

79. In Fig. 13 we have represented one of the best forms of arch for short spans; for in consequence of its being free to turn at three points A, c and B,

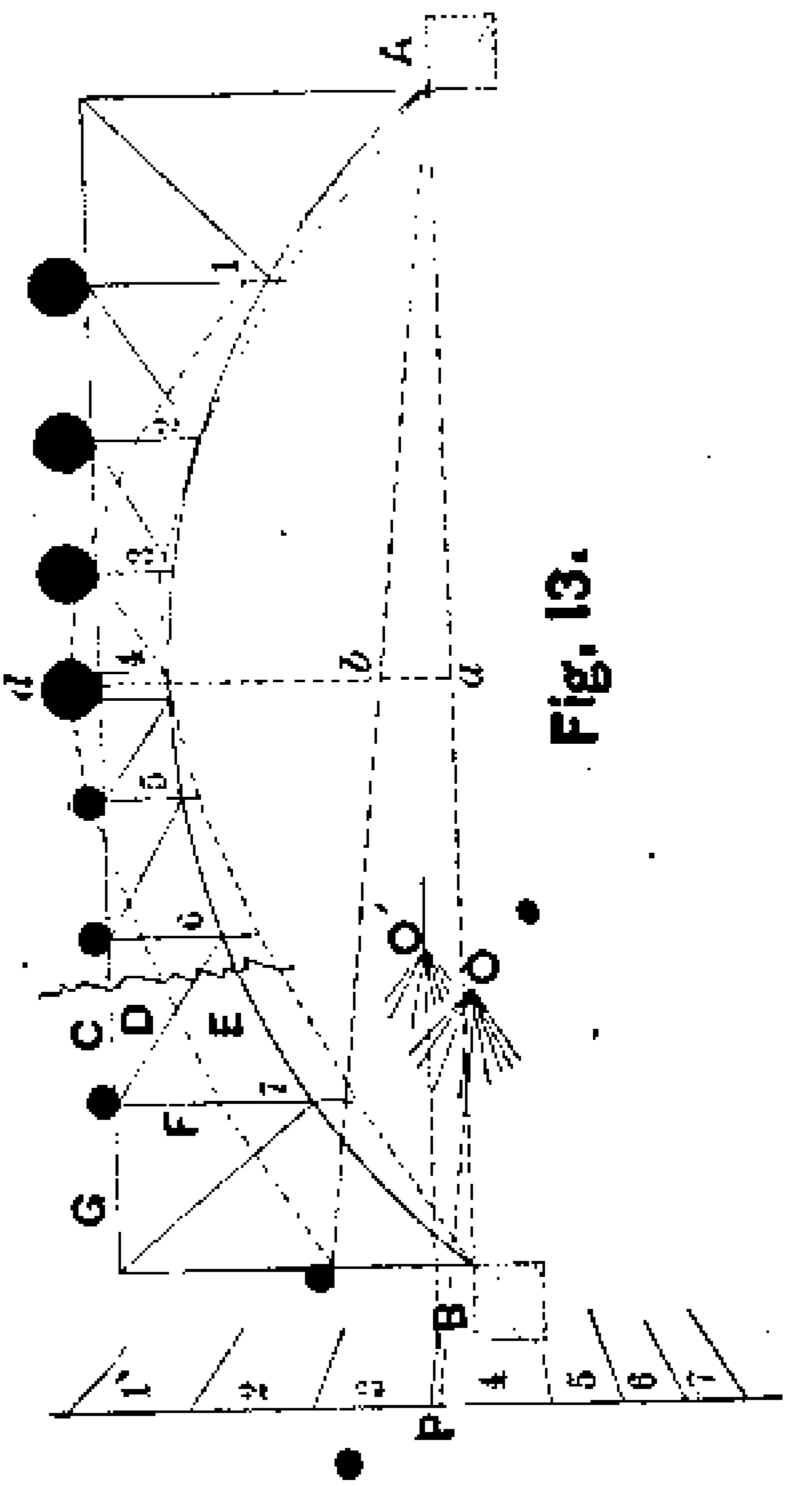


Fig. 13.

the upper chord being cut at *d*, and the arch proper at *c*, there are no strains due to change of span, temperature and elastic shortening of the arch. Let us suppose that the two halves of the arch bear at *c* only. Suppose the right half of the arch to be loaded at the apices with the weights 1, 2, 3 and 4, shown on the left, due to dead and live load; the left half with the weights 5, 6, 7, due to dead load only. With an assumed pole *o*, draw the equilibrium polygon shown by the dotted line passing above *d*. Now since there can be no bending moments at *a*, *c* and B, the actual pressure curve must pass through these points. Therefore draw a line from *o* parallel to A*b*, the closing line of the trial curve, to intersection with the load line 17, and from this point draw a horizontal, of a length equal to the old pole distance multiplied by the ratio of the ordinates $\overline{bd}$ and $\overline{ac}$, corresponding to the two curves at the center. This fixes the new pole *o*′, from whence the true curve, passing through A, *c* and B, can be drawn.

This simple construction has been used by Prof. Eddy (see Researches in Graphical Statics) for passing a curve of pressures through three given points of a stone arch, as well as for the case above.

80. The pressure curve passes near the curved member from A to c, then keeps below the arc to B, for the loads assumed.

For a single weight, as that over apex 2, the pressure curve consists of two straight lines; one drawn from B, through c, to intersection with the vertical through 2, the other drawn from this last point to A. The strains due to each weight may be found from its pressure curve, and tabulated; from whence the maximum strains that any member can ever be subjected to, from the most hurtful distribution of the load, can be ascertained.

METHOD OF FINDING THE STRAINS IN THE MEMBERS OF ANY ARCH.

81. The above figure will answer, by

way of illustration, for the method to be pursued in any arch. Thus, suppose for any arch, fixed at the ends, &c., or otherwise shaped like Fig. 13, that we desire to know the stresses sustained by pieces, C, D and E, the resultant R=ray 0′67, acting along the side of the equilibrium polygon included between the verticals at 6 and 7.

Conceive a section, as shown by the wavy line, cutting C, D and E. Suppose, now, the arch to the right of the section removed, and its effect upon the left part replaced by forces acting *opposed to the resistances* in pieces C, D and E.

The part left of the section is still in equilibrium. Now, R, acting to the right, is the resultant, in position, of the reaction and loads *left of the section;* hence it must be in equilibrium with the *forces* applied to the cut pieces, C, D and E, which forces we may denote by the same letters as the pieces to which they correspond.

Therefore, the moment of R about any point, must equal the sum of the moments

of C, D and E. Thus, take 6 as a center of moments; the moments of E and D are zero, hence the moment of C about 6 equals the moment of R, from whence C can be found.

Again, take apex DG as the center of moments, we have the moment of E equal to the moment of R about DG, from whence E follows.

We see in this case that the pieces C and E are in tension and compression respectively, since the forces C and E must act from and towards the cut pieces respectively to cause equilibrium with R.

It may be observed that when R is on the other side of the apex taken as the center of moments, that the strains caused are of an opposite character.

The strain in D can be found by taking moments about 7. The moment of R must equal the sum of the moments of C and D, etc.

Similarly, if we suppose E, F and G cut, and the part of the arch right of the section removed, including the weight at GD, supply forces E, F and G, opposed

to the resistances in E, F, G; we have these forces in equilibrium with the resultant of the external forces to the left of the section, which is now simply the reaction at B (ray 0′7 P). With 7 as a center of moments we find G; and with the apex at the left end of G, as a center of moments, knowing E, we can find F; and so we proceed through the arch.

The well known Maxwell method of diagram may also be employed in this case, as illustrated by Du Bois in his Graphical Statics.

82. When the flanges of an arch are parallel, or nearly so, as usually happens, their strains may be determined as above; but the strains in the arch members are now very easily found by decomposing R, for the section taken, into components, N and T, normal and parallel to the arch at the section. This normal component, multiplied by the secant of its inclination to the diagonal cut, gives the strain in the latter. Thus in Fig. 14, let R, acting through a, be the resultant

of all the forces to the left of the section, which must therefore be in equilibrium with the forces C, D, C′, that are opposed to the resistances of the cut pieces. Now by mechanics, the algebraic sum of the tangential components are zero; *i.e.*,

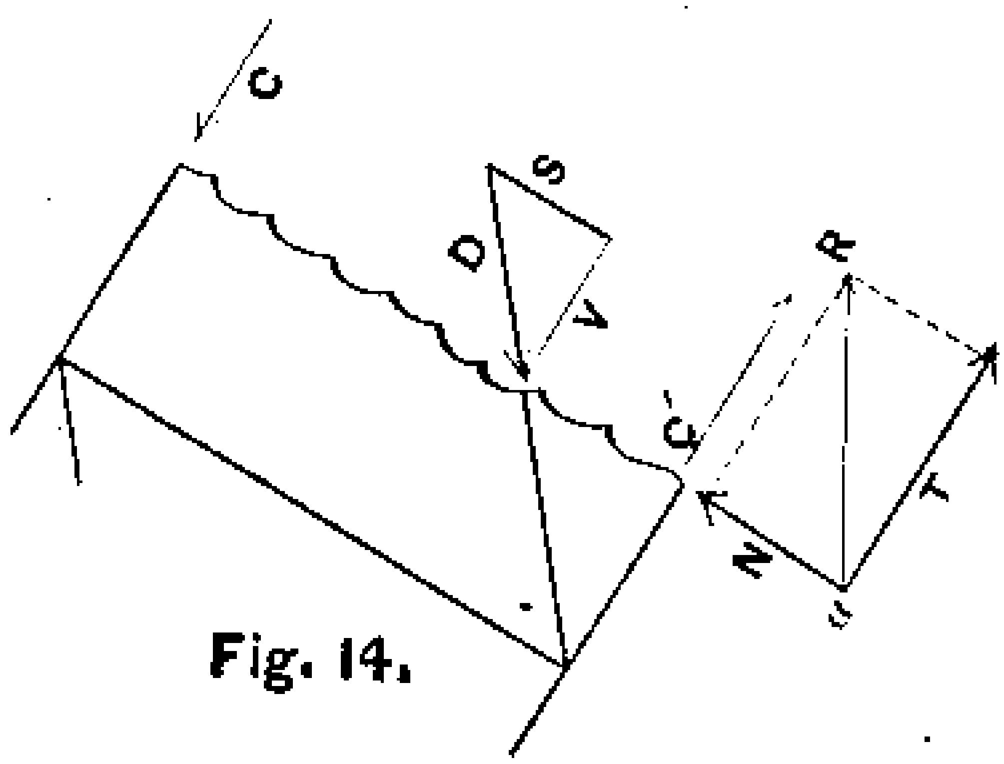


Fig. 14.

$C+V-C'-T=o$; also the sum of the normal components; *i.e.*, $S-N=o$, whence

$$S=N.$$

On multiplying N by the secant of the inclination of the diagonal to the normal, or laying off S to scale and drawing V parallel to T, we find the strain D in the diagonal.

83. The method of art. 11 applies here also; both methods applying to the solid arch with a continuous web.

The proper resultant, acting at the supposed section, must be carefully noted; thus in Fig. 13, when E, F and G are the cut pieces, the resultant left of the section is the reaction acting through B, and not the resultant, 0′67 corresponding to the next section through E, D and C, since the weight at CG is then included.

Similarly in Fig. 11, for the case of the arch fixed at one end, the resultant for a section between A and a_1, *not* including the load at a_1, is the reaction passing through O; the resultant for any section between a_1 and a_2 is ray 0″12, acting along e_1 e_2, and so on.

It is seen from Fig. 14, that when N acts upwards, that D is compression or tension, according as the top of the diagonal leans to the right or left of the normal; the reverse when N acts downwards.

84. The position of the live load that gives maximum stresses varies for each piece; so that it is only by computing the stresses caused by each single weight,

and combining the resulting positive and negative strains, that the maximum of either kind may be found. As some of the positions of the live load may be rarely experienced (see "Maximum Stresses in Framed Bridges," by the writer, for similar facts in connection with simple girders), it is, of course, an open question, whether the effect of such unusual positions of the load may not be included under some per-centage added to usual strains. In all cases, however, it is correct practice to ascertain these max. strains, in order that the piece may be designed, so that for no loading, the limit of elasticity will be passed.

UNSYMMETRICAL ARCHES.

85. The preceding principles are applicable to unsymmetrical arches, as where one end of the arch is higher than the other, etc.; only the closing line of the arch regarded as an equilibrium polygon will not generally be horizontal, but must be drawn, as that in the polygon

c due to the loading, to satisfy the proper conditions.

BRACED ARCHES.

86. The theory previously given is nearly correct for the *solid arch*, having a thin plate web (whose effects in causing a flexure is included in the term I), notwithstanding we have neglected the influence of the shearing or normal component N, in the formulæ of art 9, *et seq.*

If there are variations in E and I, for the different portions of the arch, we have seen that such variations are easily included in the graphical treatment, however difficult this may become in the strictly analytical solution. As a rule, however, the flanges, either in the solid or braced arch, are varied but little in size, except for very large spans, so that E as well as I is generally constant. But is the theory applicable to a *braced arch*, or one composed of two flanges with diagonal bracing? Since we cannot include this bracing, ordinarily, in the term I, its influence in causing bending

deflection is neglected. In fact the strains on the diagonals are caused entirely by the normal components which are disregarded. The question then is the following: Is the total change of inclination of the tangents between any two points of the arch, or the relative displacements of those points, materially influenced by the web of the braced arch?

Charles Bender, C. E., has given, No. 26 of Van Nostrand's Science Series, an analytical treatment of the case of the deflections of a straight girder (simple or continuous). From his eq. 3, p. 82, we find, on substituting the values of $\gamma_1\,\gamma_2, \ldots$, that the influence of the web in causing a change of inclination of end tangents is very small. This is approximately true for arches—flat arches especially. From eq. 4, p. 84, however, we see that angle x involving the deflections of a cantilever beam *is* influenced by the web. We should expect the same in an arch. It is easily prove by a careful construction thus: draw the

arch with diagonal bracing, then starting at any apex construct the arch as changed from the tensions or compressions of its members, finding each successive apex in turn. If the shortening or lengthening of the pieces is exaggerated, we have an exaggerated view of the deformed rib. Now, if the same construction be used when the diagonals are supposed unaltered in length, it will be found that the deformed rib differs materially in position from the first, and the more so the greater the depth of arch.

Stoney ("Strains in Girders," Plate 1) has given a similar construction for a straight truss. The difference in position of the two deformed girders, given by him in Plate 1, may be taken as a partial illustration of the bent arches given above. As a particular case, the pressure curve was assumed to pass through the apices of the lower flange, so that the change in the top flange was zero. The diagonals are also under no strain or very little. The exaggerated figure of the arch was then drawn; also

the figure when the diagonals are supposed alternately lengthened and shortened, about one-third the corresponding change in lower flange.

The depth of arch was taken at about one-seventh of the radius; the half central angle being $72\frac{1}{2}°$, the semi-arch being divided into five panels. The departure from the free end was about $\frac{1}{6}$ greater for the case where the change in the diagonals was considered—certainly a very large increase to be neglected.

It is to be observed, however, that the pressure curve is variously inclined to the neutral line, and that the influence of the web when N acts up, is the opposite of that when N acts down; so that the error made in neglecting the web's influence may ultimately be small in consequence of this partial balancing of errors.

Again, if the sections of the web members are very large compared with usual strains, their alterations in length and influence in causing deflection becomes

smaller in inverse proportion to the size of the pieces.

In the case of bridges with flat arches and small depth of arch ring, the influence of the open web diminishes greatly; but an actual construction for each case can alone determine the amount of error involved, which may thus be easily included in the factor of safety. For deep arches with small radius, however, such as some roof trusses, it is best to use a solid web; otherwise the theory hitherto given may not be applicable, and the strains become indeterminate. At any rate, it is well to test the influence of the web, for any design, by an exaggerated construction of the deformed arch in order to include the error made under a proper factor of safety.

This objection does not apply to the braced arch with three hinges, since the theory of elasticity is not involved in establishing its true pressure curve.

87. The theory of elasticity is applicable, though in different degrees, in obtaining the true pressure curve, to *all* struct-

ures resting on two supports when a real or virtual change of space is considered. Thus, consider any form of girder or roof truss that fits perfectly when laid flat on its side, not then being subjected to any strain. When erected in place and the supports removed, the tendency is to push the abutments further apart, due generally to the compression of the upper members and the tension of the lower ones If one end of the truss is on rollers, it slides, and the reactions are more nearly vertical, the smoother and more perfect the rollers or other device used. But if we suppose the abutments fixed in position, the reactions have to be determined by the condition that "the span is invariable." Even in a straight girder, where the lengthening of the lower chord is in part taken up by the curvature, the reactions are inclined slightly inwards; thus decreasing the strain in the lowest chord, whilst not adding to the strain in the upper chord. This lengthening of the span is thus on the side of safety, whether produced by

the elastic elongations of the parts or by a rise of temperature over that at which the parts are fitted.

A fall of temperature, on the other hand, may cause the reactions to incline outwards.

Now, this tendency to a change of span is much greater in some structures than others, particularly in those of the abutting class. Thus, take the very graceful "crescent roof truss" with diagonal bracing: if one end rests on rollers it will practically act as a girder having vertical reactions; otherwise it becomes a braced arch, and its pressure curve is to be determined by previous conditions. The flanges in this case not being parallel, the formulæ of arts. 9–20 no longer apply.

Bow, in his "Economics of Construction," has called special attention to the above facts. On page 86 he gives an interesting method of finding the *true reactions* of a roof truss whose parts have been changed in length, due to the elasticity of the material. Thus, to take one

case: Draw the diagram of forces for the reactions vertical, and compute the change of span, Δs, due to elasticity on the assumed sections. Next, find the change of span, $\Delta s'$, due to an assumed horizontal thrust, H. Since this change varies directly with H, we may now alter H in the ratio of Δs to $\Delta s'$, so that the total change of span is zero. The true reaction is compounded of the last value of H and the vertical reaction. Similarly, if the change of span is not zero, the final value of H must be such that $\Delta s - \Delta s' =$ assumed change of span.

It is suggested that the change of span due to elasticity can be readily found, as explained in the preceding article, from an exaggerated drawing of the deformed structure.

When the truss rests upon high, narrow abutments, their yielding is an unknown factor, so that it is well in such cases not to use a structure having much tendency to a change of span.

SUSPENSION SYSTEMS.

88. The theory hitherto exposed is equally applicable when the arches are

suspended from the supports, the strains being changed in kind only. The suspended arch with three hinges has been lately used, and is peculiarly suited to overcome that want of rigidity observable in ordinary suspension bridges, besides being readily analyzed.

ARCHED ROOF-TRUSSES.

89. The arched roof truss is more especially adapted to large spans. In this investigation, the flanges will be supposed concentric or parallel, so that the conditions of art. 9 *et seq.* apply; more especially if the web is solid or latticed, and approximately for a more open web.

When the weight of the structure, with any vertical loading, as snow, alone is considered, the analysis falls under that already given. If in addition, however, the wind is supposed to blow on one side of the structure, the case becomes one of oblique forces and is more difficult of solution.

90. If we suppose the wind to blow horizontally with a force P against a

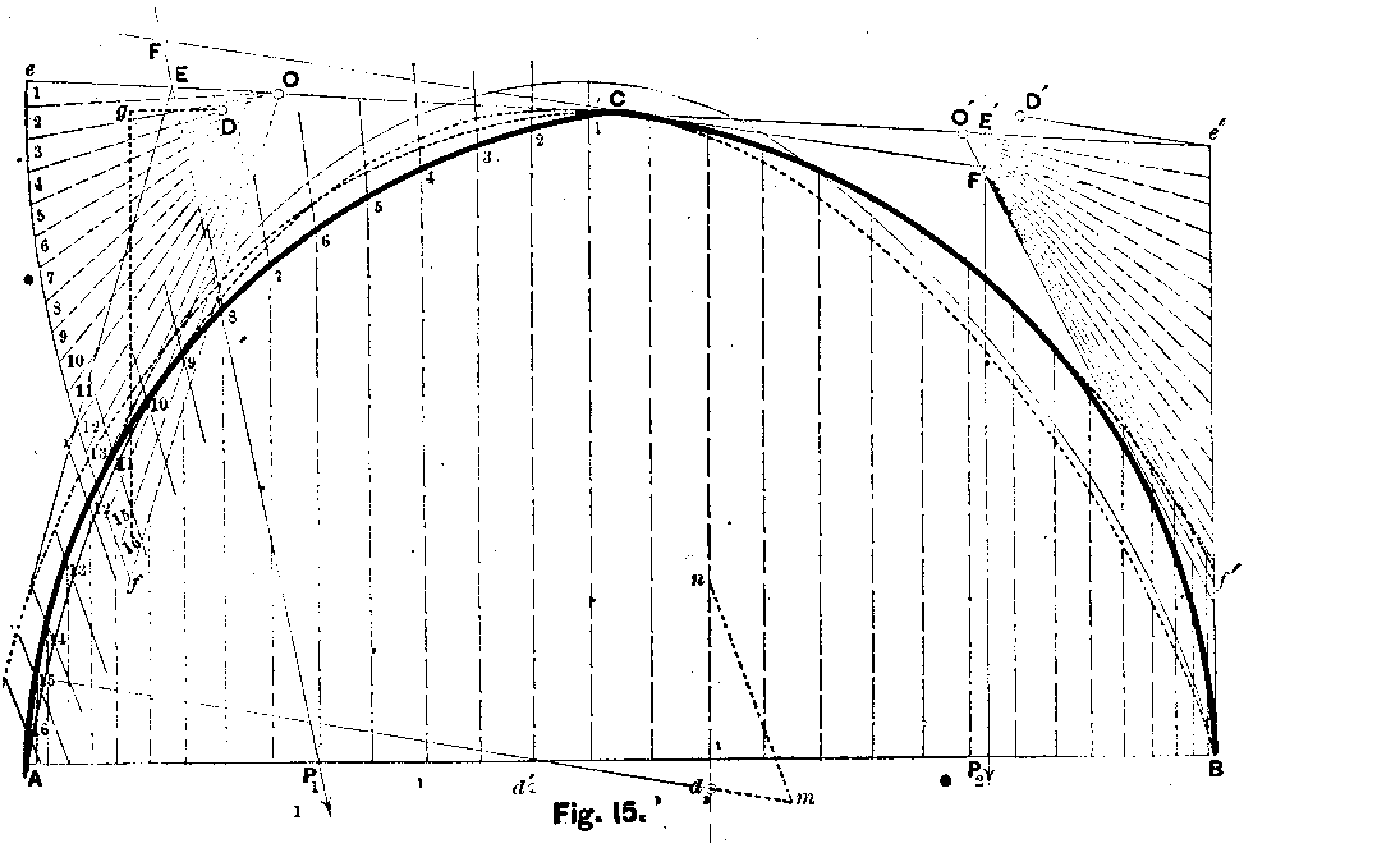


Fig. 15.

square foot of vertical plane, the experiments of Hutton go to show that the normal pressure on a square foot of roof surface inclined at an angle i to the horizontal equals

$$N = P \sin i^{1.84 \cos. i - 1}$$

More extended experiments are needed to verify this formula; but assuming it to be true, we have—taking P=40 lbs. per square foot, as the greatest intensity of the wind in a horizontal direction likely to occur, the following values of N in pounds for different inclinations of the roof surface, taken from Greene's Roof Trusses:

i	N	i	N	i	N
5°	5.2	25°	22.5	45°	36.1
10	9.6	30	26.5	50	38.1
15	14.	35	30.1	55	39.6
20	18.3	40	33.4	60	40.

For steeper slopes N is 40 lbs.

91. Let ACB, Fig. 15, represent the neutral line of a pointed roof truss; the

arc AC being described with a center d, the arc BC with a center d'. In this figure each half arc is divided into 16 equal parts and ordinates drawn through the center of each division.

Assume the slope of roof i, on each division, to be that of a tangent to the center of each division, then the force of wind on the 15th division, inclined 80° to the horizon, is 40 lbs., and it acts in the direction of the radius $\overline{15, d}$. Hence lay off $\overline{dm}$ on the radius produced equal to 40 lbs, multiplied by area of division; if $\overline{dn}$ represents the weight of one division of roof, $\overline{nm}$ is the resultant oblique force acting at the center of division 15, through which a line is drawn parallel to $\overline{nm}$. The weight of each division, including purlins, sheeting, snow, &c., does not generally act through the center of the division of the neutral line; hence it is most accurate to combine N_1 with the vertical load acting through its center of gravity; so that the oblique line just drawn will be moved slightly to

the left. The subsequent constructions are the same in either case.

92. Proceeding in this manner for each division, values of N being interpolated from the table, when necessary, we next lay off on a force diagram on the left, beginning at e, the forces 1, 2, . . . , acting at the center of divisions 1, 2, . . . , in order and parallel to their directions. Similarly, on the vertical through e', lay off the equal weights on the divisions from C to B in order.

(The forces are here laid off to a smaller scale than that to which $\overline{nm}$ was drawn, for convenience).

The lines $\overline{ef}$ and $e'f'$ represent, therefore, the intensity and directions of the resultants of the forces just found from C to A and from C to B respectively.

The position of these resultants P_1 and P_2 are found as follows: The line eCe' was drawn in the first instance passing through the crown, at about the inclination it was thought the pressure curve would have there. Hence, assuming O and O', equally distant from e and e',

as poles, draw the pressure curves for the left and right halves of the arch respectively, starting at c with the assumed inclination of the thrust there. The curve on the right is drawn as usual and needs no explanation. It lies very near the neutral line at first and then passes above it. On prolonging the last side to intersection E′ with Ce', we find the position of the resultant P_2 of the forces on the right half of the arch.

For the left side, we extend Ce to intersection with force acting at 1, then draw a line ∥ ray 12 to force at 2, then a line ∥ ray 23 to force at 3 and so on. On prolonging the last side to E, we have $EP_1 \parallel ef$ the position of the resultant $P_1 = \overline{ef}$ of the oblique forces acting on the left half of the arch. If this were the true pressure curve, this "last side" would represent the *reaction* at A, which produced to intersection E with the thrust at C gives one point of the resultant P_1 as stated.

93. It is evident that so long as the loading remains the same that the position-

tions and magnitudes of P_1 and P_2 remain the same for any pressure curve. It is now very easy to find the reaction at A in order that a pressure curve may pass through the three points A, C and B. Thus call the vertical and horizontal components of the reaction at A, V and Q respectively. On equating the moment of V with that of P_1 and P_2 about B, we find V

$$\therefore\ V.\overline{AB}=P_1 p_1+P_2 p_2,$$

calling p_1 and p_2 the length of perpendiculars from B upon P_1 and P_2 respectively.

This supposes B to be at the same level as A, otherwise Q will have a moment about B. If P_1 is decomposed into vertical and horizontal components at the level of B, the term $P_1 p_1$ may be replaced by the product of its vertical component by its horizontal distance to B. To find Q take moments about the crown of the forces to the left of it. Thus calling the length of the perpen-

diculars from C to P_1 and AB, c_1 and h respectively, we have,

$$Qh = V\frac{AB}{2} - P_1c$$

from whence Q is found.

Laying off, now $\overline{fg}=V$ and $\overline{gD}=Q$, we have D as the new pole on the left. On drawing $e'D'$ parallel and equal to eD, we have D' for the position of the new pole for the forces acting on the right half of the arch.

The direction of the pressure at C is the line FcF' parallel to eD or $e'D'$.

Starting at C we draw the pressure curve as before. It is shown by the dotted line passing through A, C and B.

It is well to test the computed values of V and Q, before drawing the pressure curve, by drawing through A and B lines parallel to fD and $f'D'$. If these lines intersect FF' at the same points F and F' with P_1 and P_2, the poles D and D' have been correctly found.

94. *The above pressure curve is the true one for the roof truss hinged at* A, C *and* B.

When the arch is not hinged at the abutments, the pressure curve will not generally pass through points at the same level at the springings. Thus suppose B of the new pressure curve to lie above A, and the point C to one side of the crown, and let it be required to pass a pressure curve through the new positions of A, B, C. First find the resultants (which call P_1 and P_2 as before) of the loads left and right of C (new positions of A, B, C, are intended in what follows). This is easily done by producing the proper sides of the equilibrium polygon to intersection, &c.

Denote the lever arms V, P_1 and P_2 about B, l, p_1, and p_2 respectively, the lever arms of V and P_1 about C, c and c_1 respectively. The vertical distances of C and B above A are h and b respectively. Taking moments now about B we have,

$$Vl = P_1 p_1 + P_2 p_2 + Qb$$

Again, taking moments of the forces to the left of C about C, we find,

$$Vc = P_1 c_1 + Qh$$

Eliminating Q from these equations, we get,

$$V = \frac{P_1(p_1h - c_1b) + P_2p_2h}{hl - cb}$$

Having found V from this equation, we can deduce Q from the preceding equation, and then proceed as before to draw the pressure curve through the assumed points A, B and C.

These formulae may also be employed in the case of a tunnel or culvert arch acted upon by oblique forces on both sides. See other formulae in "Voussoir Arches," art. 63.

95. Recurring to the pressure curve ACB, having the poles D and D′, we have the *bending moment* about any point on the neutral axis between 4 and 5, say, equal to ray D 45, measured to the scale of force multiplied by the perpendicular distance from the point to the resultant acting along the side 45 of the *pressure line.*

This moment is also equal to the horizontal component, H′ of ray D45, multiplied by the vertical distance v from the

point to the resultant. This is evident, if we decompose the resultant at a point of its line of action vertically over the point taken in the neutral axis, into vertical and horizontal components. The latter alone causes a moment about the point equal to $H'v$ as stated.

It is well to note that v is not measured necessarily to the pressure curve, unless the resultant happens to act along the side vertically over the point. At a point where a force acts, as the one marked 4 of the neutral axis, the bending moment may be found by taking the moment of the resultant acting either side of the force. In this case if the resultant (=ray D45) to left of force is taken the moment$=H'v$, v being the vertical distance above 4 to the side of the pressure line between forces at 4 and 5, produced.

As the oblique forces are inclined inwards, it is seen that if we measure the distances v from 1, 2, . . . to the pressure line as usual, that when the pressure line is above the neutral line, H' must be taken as the horizontal component of the

resultant acting just to the *right* of the force acting at the point taken; otherwise, when the pressure line is below the neutral line, H' corresponds to the resultant acting just to the left of the force.

Thus at 4, use the ray D34, at 13, the ray (D13, 14), in evaluating H', v being measured vertically from 4 and 13 to the pressure line. As usual, v is + when above the neutral line, — below it, in finding ΣMy, etc.

96. For *the arch hinged at the ends, but continuous at the crown*, we have the conditions (E, I and s being constant) that the span is invariable,

$$\therefore \Sigma My = O$$

The moments M to the left of C are of the type $H'v$ H' and v both being variable; to the right of C, $M=Hv$, v alone being variable, H representing the constant pole distance from D' to force line $e'f'$.

The ordinates y are, of course, measured from AB to the neutral line of the rib; and in the summation, v will be

taken as plus when laid off above the neutral line, minus below it, since the moments M have different signs on opposite sides of the neutral line.

97. It is seen by inspection that the pressure curve passing through A, B and C must be raised at C to satisfy the eq. $\Sigma My=O$. Hence, a trial pressure curve was drawn, passing through A and B and a point about $\frac{1}{20}$ of an inch above C. Now measure to some scale (40 was used) the values of H′, v and y for each point 1, 2, . . . , for both halves of the arch and compute $\Sigma My=\Sigma H'vy$. It is found to equal in this case -1000. The curve should be raised slightly at the crown.

Now if we suppose applied at A a horizontal force Q′, and at B an equal force Q′, opposed to the other, we do not disturb equilibrium. The moment M′ at any point of the rib whose ordinate is y, due to Q′ is Q′y.

Now, if for an assumed thrust Q, we find that $\Sigma M'y=\Sigma Q'y^2=+1000$, it follows that by combining Q′ with the re-

actions previously found at A and B and drawing a new pressure curve, that we shall then find the resulting $\Sigma My = O$, since the moments caused by the resultant reaction must equal the sum of the moments caused by its components. To find Q', we have simply to compute $\Sigma(y^2) = 1500$ in this case; then since

$$Q' \Sigma y^2 = \Sigma My \text{ (first found)}$$

it follows that $1500Q' = 1000 \therefore Q' = \frac{2}{3}$.

This acts outwards in this case, since the curve has to be raised; hence lay off $Q = \frac{2}{3}$ from D in the direction Dg to find the true pole for the force polygon at the left, from whence the new position of D' is found as before, and the pressure curve drawn as usual.

If the first ΣMy had been positive, then Q would act as a thrust, and the pole distance would be increased.

It is perceived that the vertical components of the reactions remain unchanged by the alteration in Q, as in fact follows from the preceding formula for computing V, in which no term involving Q occurs.

The true pressure curve, for the arch hinged at the ends, is shown by the full line in the figure, passing through A, B and a point nearly $\frac{1}{10}$ inch above C.

It is seen that this case offers no difficulty.

98. A tentative solution has been given by Wm. Bell, C. E. in his paper on "Rigid Arches, &c.," (see Van Nostrand's Magazine for May 1873), where the reader will find another method given of finding the reactions for a pressure curve assumed to pass through three points; also the graphical computation of the wind forces, according to the law of "the square of the sines."

In drawing the pressure curve for the arch "fixed at the ends," Mr. Bell has throughout neglected the condition $\Sigma Mx = o$, which, thus far, invalidates his results. The reader is referred to his paper for the case of the "rigid arch braced" on the supposition of hinged joints; which, even for straight rafters, represents the true *pressures* and *moments* at any point of the rafter for con-

tinuous loading, which is not so clearly seen when, as is usual, the loads are supposed concentrated at the apices. The writer has also illustrated this point in Science Series No. 12, Fig. 17. If the loads are supposed concentrated at the apices, the longitudinal strain at the *middle* of the division of the rafter only isthe same as that found from the consideration of continuous loading.

The bending moment there is, of course, the same, whether found graphically or computed for an inclined beam supported at the ends and continuously loaded.

Even for the case of curved rafters, the longitudinal strains, acting in the straight line joining two apices, due to the supposition of the loads being concentrated at the apices, may be found by the simple diagram of forces. The total strain at the center of the division of the rafter is then due to this strain, acting with its leverage to the neutral line, and the moment caused by the continuous loading on the part of the rafter considered "supported at the ends."

The supposition of hinged joints is not exactly realized for curved or straight rafters, however.

99. *Arched Roof Truss, fixed at the ends, continuous at the crown.*

The strains due to vertical loads may be found as hitherto illustrated in Fig. 6. An *exact* solution of the strains caused by the wind, is obtained by considering the wind force acting at each point *separately*. The case then becomes identical with that of an unsymmetrical arch acted on by a single vertical load, the arch being supposed tilted up at one end until the wind force, at the point taken, is vertical. The ordinates y must now be drawn parallel to the direction of the wind force (some of them may pass outside the arch), to the line joining the ends of the arch, which is taken as the axis of x.

The conditions, $\Sigma M=o$, $\Sigma Mx=o$, $\Sigma My=o$, must be fulfilled, as is readily shown by the reasoning of arts. 12, *et seq.*; the first condition indicating that the tangents at the ends are fixed in po-

sition, the second and third, that the displacements of one end, relatively to the other, in directions perpendicular to x and y respectively, are zero.

The construction is now proceeded with as shown by Fig. 6, except that the closing line k_1k_2 of the arch is no longer parallel to the axis of x, but must be determined precisely as shown for curve p.

100. The general demonstration of art. 34 is especially useful here in indicating precisely the steps to take, as some of the ordinates, passing outside of the arch, appear to produce abnormal features.

Now it will doubtless be considered too tedious in practice to consider each wind force separately, hence a method that will give the strains due to wind force and vertical loading both, at one operation, is especially desirable in a practical point of view. There are considerable difficulties in the way of an exact solution; but it is believed that the method given below, which leads by one or two approximations to a result very near

the truth, is sufficient for the needs of the practitioner, especially in view of the fact that the theoretical conditions are not often realized in putting the structure together.

These conditions are—that the parts of the arch fit perfectly when under no strain, the fitting being tested by putting the parts together so that the arch lies horizontally, or on its side.

Again, the abutments should be connected with the arch by bolts; and should be so heavy that the bending moments at the feet of the arch will cause no appreciable rocking in them.

It is likewise a known fact that the wind pressure is greater as we ascend, especially if there are obstructions to its course near the base of the structure. Mr. Bell, in his attempted analysis of the roof of St. Pancras Station, has neglected the wind force near the abutments; possibly from the cause mentioned, though mainly, I presume, in consequence of the increase of section towards the abutments due to vertical walls, with

a sort of spandrel filling between them and the arch proper, as well as the introduction of doors and windows in the part considered.

By neglecting the wind force on the lower part of the arch in the construction below, we are led to a more rapid approximation to the true pressure curve, so that it is in the line of simplicity.

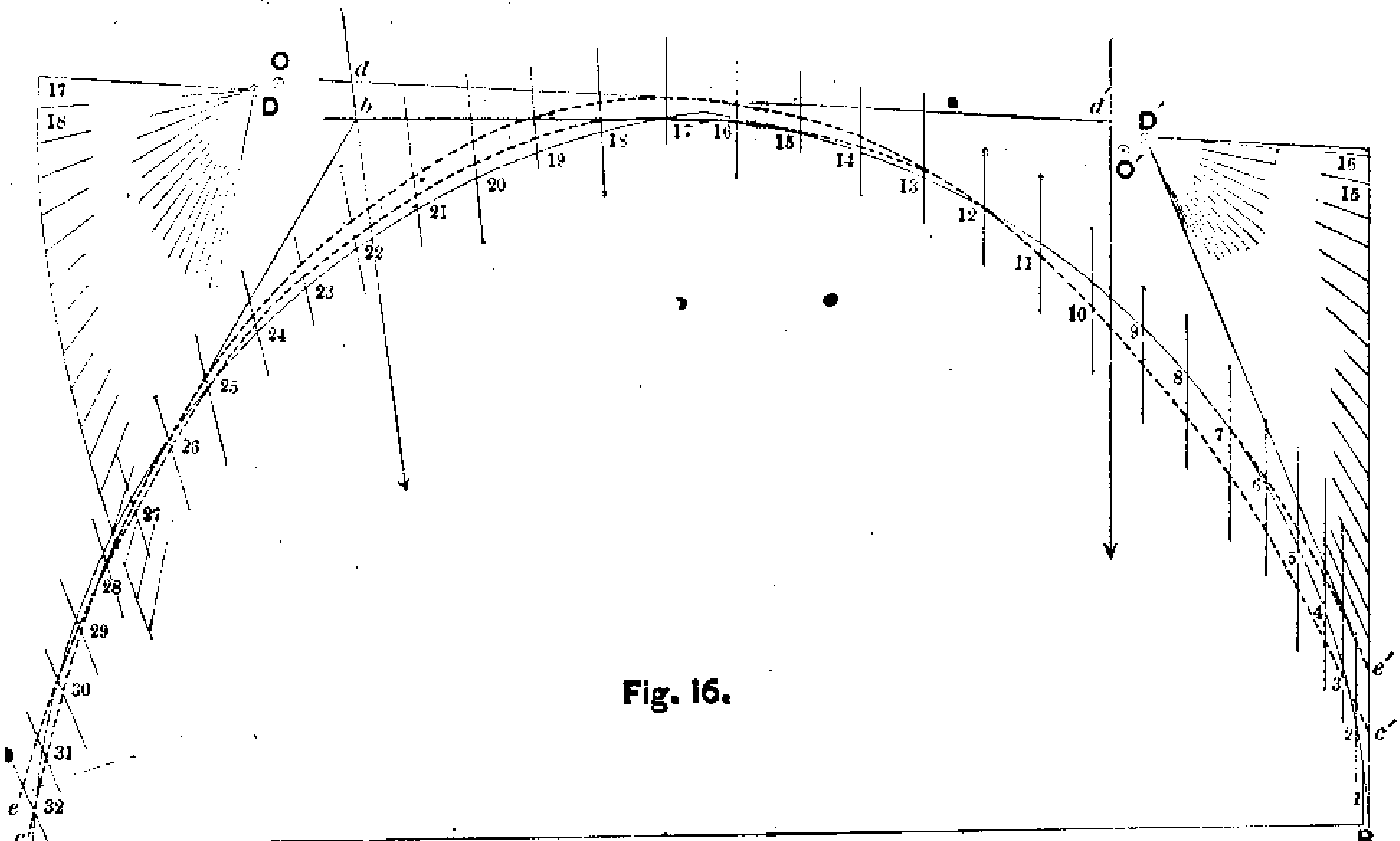
101. Let us take A (Fig. 16, *see plate*) as the origin of co-ordinates, x horizontal, y vertical. The true pressure curve for the arch fixed at the ends must then satisfy the conditions,

$$\Sigma \mathrm{M} = o, \ \Sigma \mathrm{M}x = o, \ \Sigma \mathrm{M}y = o.$$

These conditions may also be written (art. 95),

$$\Sigma(\mathrm{H}'\nu) = o, \Sigma(\mathrm{H}'\nu x) = o, \ \Sigma(\mathrm{H}'\nu y) = o.$$

Now let us draw a trial pressure curve as near the true one as possibly can be estimated. There is no guide from previous examples, as this case has never been correctly solved hitherto, but we may infer from the preceding equations, that the true pressure curve must cross the cen-



O
D
d
b
a
d′
D′
O′
17
18
15
16
1
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21
22
23
24
25
26
27
28
29
30
31
32
e
e′
c′
B

Fig. 16.

ter line several times, and, moreover, must depart from it more on the left (H' being less) than on the right half of the arch.

The first trial curve was drawn through e, e' and a point slightly below the crown, as shown by the dotted line (on divisions 7 to 12 the curve coincides with the neutral line of the arch), the poles being O and O'. Call the constant pole distance for the right half of the arch H, the horizontal component of the resultant at any point of division on the left half of the arch H', and the ratio of H' to H, θ.

If we denote the vertical ordinates from AB to the arch and trial pressure curve, by y and y' respectively, we have at a point on the left of the center,

$$M = H'\nu = H\theta(y' - y)$$

It is sufficiently near to measure y' to the equilibrium polygon simply (and not to certain resultants produced as explained in art. 95) for the first construction.

102. The first two conditions above become,

$$\Sigma\theta(y'-y)=o,$$
$$\Sigma\theta(y'-y)x=o;$$

the summation being extended over the entire arch, θ being, of course ity for the right half of the arch.

Next diminish the ordinates y and y' at each point of division on the left of the arch to θy and $\theta y'$, using the value of θ corresponding to the point, and lay off the altered curves as shown in Fig. 17, the right half of the arch, &c., being the same as before. The values of H′ are readily found by measuring, to scale, the horizontal distances from the points of division of the force line to a vertical drawn through O. The altered lengths may be easily found by a slide rule or by the usual ratio lines.

103. Let us denote the ordinates from a straight line k_1k_2 (Fig. 17) to the altered curve of the neutral line by a, and draw k_1k_2 so that $\Sigma a=o$, $\Sigma ax=o$. The construction is similar to that used in finding the closing line mm' of polygon p in Fig. 6.

If we denote the ordinates from k_1k_2

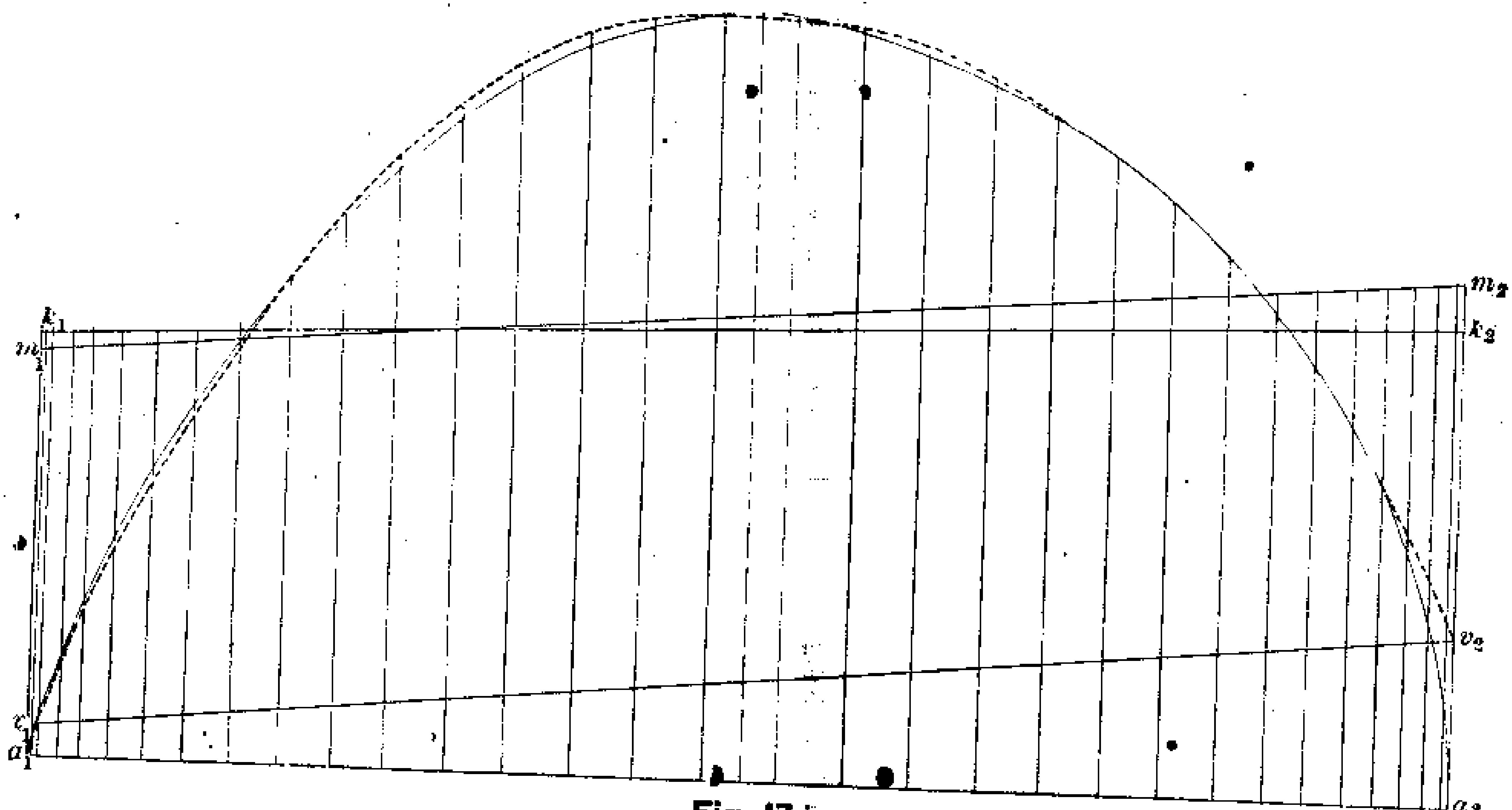

Fig. 17.

to the pressure curve by c, the above equations become,

$$\Sigma(\theta y'-\theta y)=\Sigma(c-a)=o$$
$$\Sigma(\theta y'-\theta y)x=\Sigma(c-a)x=o$$

Now since $\Sigma a=o$, $\Sigma ax=o$, it follows that when the pressure curve is adjusted to its right position that $\Sigma c=o$, $\Sigma cx=o$ also. Therefore, calling the ordinates from a straight line m_1m_2 to pressure curve, c', m_1m_2 being so drawn as to satisfy the conditions $\Sigma c'=o$, $\Sigma c'x=o$, we now conceive m_1m_2 placed over k_1 k_2 and the ordinates c' laid off above and below k_1k_2 to the true position of a pressure curve c, whose ordinates now satisfy the conditions $\Sigma(c-a)=o$, $\Sigma(c-a)x=o$.

104. We have now introduced one approximation, viz: that the ordinates of an equilibrium polygon, due to oblique forces in part, are unchanged in length by the particular position of the pole, so that it lies in the same vertical. This would be correct if the vertical *reactions* at the abutments of a supposed girder, acted on by the same external forces, were always found in the same

vertical, since the bending moment at any point is then *constant* and equal to $H'y''$, y'' being the ordinate from the closing line to the curve. It is evident that since H' is constant, that y'' must also remain the same for any pressure curve drawn with the same pole distances.

In this case, however, the point *e* will move along the inclined force at 32 (Fig. 16), as the pole *o* is moved up or down vertically, so that the vertical component of the reaction will not remain in the same vertical, therefore the moment is not constant. Still the result will not vary much from the truth, and would be exact, so far, if the wind forces near A were discarded.

105. We come now to the third condition, $\Sigma My = o$

$$\therefore \ \Sigma\theta(y'-y)y = o = \Sigma(\theta y' - \theta y)y = o$$

$$\therefore \ \Sigma(c-a)y = o,$$

or $$\Sigma cy = \Sigma ay,$$

c representing the ordinates from $m_1 m_2$ to dotted curve Fig. 17, *a* the ordinates from $k_1 k_2$ to curve *a* Fig. 17.

Find Σcy and Σay as in Fig. 7. This construction is readily made on Fig. 17, by drawing horizontals through the tops of the ordinates y for both halves of the arch, laying off the ordinates c and a as forces on a horizontal, and proceeding exactly as in Fig. 7. If the above equality does not hold, change the lengths c in the ratio of Σcy to Σay to cause the equality, and lay off the new values of c above or below $k_1 k_2$ (the ultimate position of $m_1 m_2$) at several points. The curve so drawn crosses curve a at points corresponding to those marked 12 and 26 in Fig. 16, c' being a third point in the proposed pressure curve, which may now be drawn through c' 12, and 26 of Fig. 16, as shown by the dotted line passing a little above A and the crown of the arch, and through the three points mentioned. This curve is described with the poles D and D′, and is evidently very near the true one, since the pole distances have been but slightly changed. If, starting with this curve, with the new values of H′, &c., we repeat the preced-

ing construction, it is plain that the resulting pressure curve will be still nearer the exact one, and will probably differ from it by an inappreciable amount, practically considered. On this small scale it is of course useless to make the second construction. The result is readily tested from the given conditions.

The first construction would be the correct one if the ordinates c could all be changed in the same ratio due to a change of pole distance (as for vertical forces), but this not being exactly true for the left half of the arch, a change in pole distance altering the ratios θ likewise, so that the positions of k_1k_2 and m_1m_2 would not remain the same, it is evident that the true pressure curve is not exactly found. The smaller the wind forces, compared with the vertical loads, the nearer the first approximation to the truth. It would seem that the best result is obtained, by passing the final pressure curve through the points of intersection (12 and 26) of curve c with a Fig. 17, m_1 m_2 coinciding with k_1k_2, as was done in this case.

106. The position D of the pole may be found very readily in a tentative way thus: let it be required to pass a curve of pressures through c', 26 and the point over the crown shown by the upper dotted line. By producing the sides $\overline{16\text{-}17}$ and $\overline{26\text{-}27}$ of any pressure curve to intersection b Fig. 16, and drawing a line parallel to the resultant of the forces from the crown to 26, as obtained from the force line on the left, we have this resultant, which pases through b, determined in position and direction.

Now draw some line dd', supposed to have about the position of the side of the equilibrium polygon at the crown, to intersection d and d' with the resultant just drawn and the resultant of the loads to the right of the crown. Then through the lowest point of the force line on the right draw $\overline{e'\mathrm{D}} \parallel \overline{c'd'}$, on the left draw ray $\overline{26\text{-}27} \parallel \overline{26.d}$.

Now from the tops of each force line draw lines parallel to $\overline{dd'}$. If these lines intersect the rays previously drawn, in points D and D′, the distances cut off

being the same, then the line dd' was correctly located. Two or three trials generally suffice to establish the poles correctly. To pass the curve through 12 26 and c', we find the resultants either side of 12 and proceed as before. It will be found best, in this case, to lay off the forces on *one* force polygon, in place of two as hitherto. It may be observed, however, that in a large drawing, especi ally, it is a great convenience to have the force polygons, as in the figure, directly over the part of the arch to which they apply.

107. The method of drawing the pressure curve detailed above possesses the merit of simplicity, but it is not so accurate as the one given in "Voussoir Arches," art. 67, since any error made—and some error is incident to any construction—is carried on, which is not the case in the following method.

At each point 1, 2, . . . , 32, resolve the load into vertical and horizontal components; find the moments about the crown of each component. Now if we consider

any number of consecutive loads from the crown towards either abutment, we have only to divide the sum of the moments of the vertical components by the sum of those components to find the distance to the resultant of the vertical components from the crown, which lay off. Similarly find and lay off the vertical distance below the crown of the horizontal components. Now on drawing a vertical through the first point found and a horizontal through the second, their intersection gives a point in the resultant of all the loads acting on the part considered, which can now be drawn in position parallel to the proper direction taken from the force diagram.

Thus we may find the resultant of all the forces acting from the crown to 26 inclusive, as shown by the arrow passing through b.

If the direction of the thrust dd' acting at the crown is given, as well as its position from its intersection d with the resultant just found, we have simply to

draw a line ‖ ray D 26, 27, to find the center of pressure at 26, or more accurately speaking, about midway between 26 and 27. The same method applies to each division in turn. On the right half of the arch the construction is simplified, since here we only have vertical forces.

If this method is not pursued, then the positions of P_1, P_2, etc. (Fig 15), should be found by at least two independent constructions, whence the true positions of the centers of pressure at the abutments may be at once found, from an assumed thrust at the crown, thus affording a check upon subsequent constructions.

108. It is evident from the considerations of art. 35 *et seq.* that the pressure curve of an *underground voussoir arch* is identical with that for the solid arch, when no joints tend to open and the mortar is thin and hard. Since in this case there are horizontal forces on both sides of the arch, the pressure curve will depart from the center line about the

same distance on either side of it at the points of maximum departure.

109. When EI is not constant for the entire arch, the term θ may be taken to represent $\dfrac{H'}{EIH}$ (see art. 41), when the construction will proceed as before, the proper values of θ being computed for each point of division of the arch.

The strains due to change of span, temperature, etc., are readily and correctly computed as hitherto explained. It is evident, therefore, that if the arched roof truss is built in accordance with the hypotheses, that the strains, due to whatever causes, may all be found.

110. It is seen from the foregoing how well adapted the graphical method is to the investigation of the strength and stability of every form of arch—many forms being almost intractable by ordinary analysis—presenting likewise the great advantage of keeping prominently before the worker the very hypotheses upon which the theory is built.

If the writer has added materially to

the knowledge on the subject of the "Solid or Braced Elastic Arch," or has aided the student in its comprehension, he feels amply repaid for the time spent in developing the foregoing theory.